Magnetorotational instability in a collisionless plasma with heat flux vector and an isotropic plasma with self-gravitational effect

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The linear stability of a differentially rotating magnetized plasma is analyzed in the collisionless approximation along with heat flux vector. The dispersion relation is derived and simplified cases are discussed with instability criteria presented. Anisotropic pressures are shown to not only alter the classical instability criterion but also induce new unstable regions. The shear rotating instability in a collisionless magnetized plasma with a scalar kinetic pressure in the presence of self-gravitational effect is then considered. Three cases are discussed specifically according to the general dispersion relation. The effects of Jeans term and compressibility on the local shear instability induced by differential rotation are examined and the analytic instability criteria are presented. © 2011 American Institute of Physics. [doi:10.1063/1.3641969]

I. INTRODUCTION

The magnetorotational instability, referred to as MRI hereinafter,1,2 plays a crucial role in driving plasma turbulence in accretion disks and is believed to be responsible for angular momentum transport, which solves a long-standing puzzle how the materials in the accretion disks fall inward to feed the stars or the black holes in the center. After the seminal monograph of Balbus and Hawley3 in 1991, MRI has attracted more and more attention in terms of theoretical analysis, numerical examination, and experimental observation by taking into account many complex and realistic assumptions such as resistivity,4 viscosity,5 the Hall term,6 dusty grains,7 and neutrals8 to investigate how the non-ideal mechanism affects the linear and non-linear properties of MRI. This is due to MRI’s important applications on the astrophysical objects such as magnetized accretion disks,9 protoplanetary disks,10,11 protostellar and stellar radiative zones,12 core-collapse supernovae,13 protoneutron stars,14 and so on (see reviews, for example, Refs. 15–18). By virtue of Taylor-Couette flow configuration, Ji et al.19 have conducted a laboratory experiment showing that the MRI appeared to be the only plausible source of accretion disk turbulence, even in cool disks. MRI is a kind of magnetohydrodynamic (MHD) instability driven originally by a differential rotation, which also has been proven to play an important role in the tokamak plasmas concerned with the tearing modes (TMs),20,21 resistive wall mode,22 Alfvén resonance,23 resistivity-gradient-driven turbulence,24 and so on.

MRI and anisotropic heat conducting effects are originally combined together by Balbus25 when the mean free path between particle collisions is much greater than the particle Larmor radius in a magnetized plasma, i.e., $\omega_c \tau \gg 1$, where $\omega_c$ is the gyro frequency and $\tau$ is the mean collision time, so that the heat is restricted to being transported primarily along the magnetic force lines.26 In such a weakly collisional plasma, the heat transporting is anisotropic whereas the plasma thermal pressure can still be assumed to be isotropic. The linear MRI of a collisionless and weakly magnetized plasma was investigated by Quataert and his coworkers.27 They found that the growth rates of MRI could differ significantly from those calculated in the framework of MHD model by using fluid equations including effects of kinetic Landau damping developed by Snyder et al.28 to determine the evolution of the perturbed pressures. Sharma and Hammett29 then studied the transition from the collisionless kinetic regime to the collisional MHD regime and figured out the dependence of the MRI growth rate on collisionality with the aid of a kinetic closure scheme for a magnetized plasma which included the effect of collisions via a Bhatnagar-Gross-Krook (BGK) operator.

In a magnetized collisionless plasma, the cross-field and longitudinal kinetic particle temperatures differ from each other, thereby making the full kinetic equations for such a plasma quite laborious. Some simplified fluid descriptions have been proposed (see, for example, Refs. 28, 30, 31). The well-known CGL model is one of the fluid equations derived by Chew, Golderberg, and Low in 1956.32 Much work has been conducted on by using the CGL equations neglecting heat flux vector or the modified CGL model including heat flux vector to investigate different kinds of instabilities, viz., “hose” and “mirror” instability (see Refs. 33 and 34 and references therein). Reference 35 describes the linear stability in a thin differential rotating disk with equilibrium pressure tensor and Hall current term by using double adiabatic equations of state and considering the effect of radiation heat loss in a scalar pressure case. The effects of uniform rotation, finite electrical resistivity, electron inertia, and Hall current on the self-gravitational instability of anisotropic pressure plasma with generalized polytropic laws were reported by Prajapati et al.36

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In the present work, we consider the MRI in a collisionless magnetized plasma with the help of the one-fluid equations including anisotropic kinetic pressure and heat conducting effect by following Refs. 30 and 37. We then study the MRI in a collisional plasma with a scalar kinetic pressure by taking into account the self-gravitational effect. The scalar pressure effect on the MRI is described in detail and the analytic instability criteria are derived. The present paper is organized as follows. The basic equations of the MHD model including pressure tensor and heat flux vector in a collisionless magnetized ideal plasma are presented in Sec. II. The linear stability of anisotropic pressure with heat conducting effect is analyzed in Sec. III in the absence of self-gravity. The dispersion relation (DR) of MRI in a collisional plasma with self-gravitational effect is derived in Sec. IV and different limiting cases are discussed. Finally, Sec. V is the conclusion.

II. BASIC EQUATIONS

The basic set of MHD equations in a collisionless magnetized plasma with anisotropic heat flux is described in the following. The plasma motion equation is

\[ \rho d_t \vec{v} = -\nabla \cdot \vec{P} + \frac{1}{\mu_0} \left( \nabla \times \vec{B} \right) \times \vec{B} - \rho \nabla \Phi \]

and the conversation equation reads

\[ \partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0. \]

The ideal magnetic frozen-in condition is

\[ \partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B}), \]

where \( \rho \) is the plasma mass density, \( d_t = \partial_t + \vec{v} \cdot \nabla \) is the convective derivation, \( \vec{v} \) is the fluid velocity, \( \vec{B} \) is the magnetic field, \( \Phi \) is the gravitational potential, \( P \) is the anisotropic thermal pressure tensor defined as \( \vec{P} = P_\perp \mathbb{I} + (P_\parallel - P_\perp) \hat{b}\hat{b} \) in which \( \mathbb{I} \) is the unit tensor, \( P_\perp \) and \( P_\parallel \) are the components of pressure parallel and perpendicular to the magnetic field, respectively, and \( \vec{b} = \vec{B}/B \) is the direction of the magnetic field. The evolution of the parallel and cross-field pressures is determined by

\[ \frac{d}{dt} \frac{B^3}{\rho^2} \left( \frac{B^2 P_\parallel}{\rho^3} \right) = -\vec{b} \cdot \nabla q_\parallel - (2q_\perp - q_\parallel) \frac{1}{B} \vec{b} \cdot \nabla B, \]

\[ \rho B \frac{d}{dt} \left( \frac{P_\perp}{\rho B} \right) = -\vec{b} \cdot \nabla q_\perp + 2 \frac{q_\perp}{B} \vec{b} \cdot \nabla B, \]

respectively, in which \( q_\parallel = q_\parallel \vec{b} + q_\parallel \) is the inhibited heat flux, whose dynamic behavior is governed by

\[ \frac{d}{dt} \left( \frac{q_\parallel}{\rho^2} \right) = \frac{P_\parallel}{\rho^2} \vec{b} \cdot \nabla \left( \frac{P_\parallel}{\rho} \right), \]

\[ \frac{d}{dt} \left( \frac{q_\perp}{\rho^2} \right) = -\frac{P_\perp}{\rho^2} \vec{b} \cdot \nabla \left( \frac{P_\perp}{\rho} \right) - \frac{P_\perp}{\rho^2} (P_\perp - P_\parallel) \vec{b} \cdot \nabla \ln B. \]

The set of equations above is coupled with the Poisson equation for the gravitational potential \( \Phi \) via \( \nabla^2 \Phi = 4\pi G\rho \) to enclose the system, where \( G \) is the gravitational constant.

We consider an axisymmetric plasma cylinder described by the standard cylindrical coordinates \((r, \theta, z)\), where \( r \) is the radial coordinate, \( \theta \) is the poloidal coordinate, and \( z \) is the longitudinal coordinate. The plasma is assumed to rotate in the azimuthal direction with an angular velocity \( \Omega = \Omega(r) \); thus, the equilibrium velocity is \( \vec{v}_0 = (0, r\Omega, 0) \). The equilibrium magnetic field \( \vec{B}_0 \) along the \( z \) direction is assumed to be homogeneous. The equilibrium condition determined by the motion Equation (1) is

\[ P_\perp = -\rho_0 (r - r\Omega^2), \]

where \( \rho_0 \) and \( P_\perp \) are the unperturbed plasma density and cross-field pressure, respectively. The prime denotes the radial derivative hereinafter. The gravitation acceleration \( \vec{g} \) is defined as \( \vec{g} = -\nabla \Phi \). Hence, the gravitation acceleration in the equilibrium state is \( \vec{g} = -g\vec{r}/r \) with \( g = \Phi_0^' \). In the equilibrium state, the gravitational potential is mainly from the central star with mass \( M \), and the self-gravitational force of the plasma mass is ignored, leading to \( \vec{g} = -GMr/r^3 \). Consequently, one finds \( \nabla^2 \Phi_0 = 0 \). In a Keplerian system, there is \( P_\perp = 0 \) and then \( g = r\Omega^2 \). As a result, \( \Omega = (Gm/r^3)^{1/2} \) is proportional to \( r^{-3/2} \).

III. ANISOTROPIC PLASMAS WITHOUT SELF-GRAVITATION

A. Dispersion relation

In this section, the self-gravitational effect is disregarded. In the cylindrical coordinates, the perturbed velocity is \( \vec{v}_i = (\vec{v}_r, \vec{v}_\theta, \vec{v}_z) \) and the perturbed magnetic field is \( \vec{B}_i = (\vec{B}_r, \vec{B}_\theta, \vec{B}_z) \). The perturbed density, pressure, and heat flux are \( \vec{\rho}, \vec{P}_\perp, \vec{P}_\parallel, \) and \( \vec{q}_\parallel \), respectively. The perturbed profile is assumed to be axisymmetrical and proportional to \( \exp(-i\omega t + ik_z z) \), where \( \omega \) is the wave frequency and \( k_z \) is the parallel wave number. In this case, the perturbed magnetic field is determined by Eq. (3) thus giving

\[ \vec{B}_r = -\frac{k_z B_0}{\omega} \vec{v}_r, \]

\[ \vec{B}_\theta = -i \frac{k_z B_0}{\omega} \frac{d\Omega}{dr} \vec{v}_r - \frac{k_z B_0}{\omega} \vec{v}_\theta, \]

\[ \vec{B}_z = -i \frac{B_0}{\omega} \vec{D} \vec{v}_r, \]

On the other hand, the perturbed conservation equation (2) yields the perturbed density in terms of \( \vec{v}_i \) and \( \vec{v}_r \),

\[ \vec{\rho} = \frac{\rho_0}{\omega} (-i\vec{D} \vec{v}_r + k_z \vec{v}_z), \]

where the operator \( \vec{D} \) is defined as \( \vec{D} \vec{f} = \partial_x f + f/r \). The entropy equation (4) has the perturbed form as

\[ [2P_\parallel (\omega - 2q_\perp - q_\parallel)k_z] \vec{B} = B_0 k_z \vec{q}_\parallel - \omega B_0 \vec{P}_\parallel + 3\omega B_0 S^2 \vec{\rho}, \]

where \( \vec{B} = \vec{B}_0 \cdot \vec{B}_i/B_0 \) and \( S^2 = P_\parallel/\rho_0 \). Note that \( \vec{B} \) here is defined as \( \vec{B} = \vec{B}_0 \), the perturbation to the magnitude of the magnetic field, rather than \( B_1 \), the magnitude of perturbed
magnetic field $\vec{B}_1$. That is, $\vec{B} = \vec{B}_2$. Substituting in $\vec{B}_2$ and $\rho$ from above,

$$i\left[P_\parallel - (q_\parallel - 2q_\perp) \frac{k_x}{\omega} \right] \vec{D} \vec{v}_r = k_x q_\parallel - \omega P_\parallel + P_\perp k_x \vec{v}_r.$$  \hspace{1cm} (12)

Based on Eqs. (9) and (10), the perturbed forms of Eqs. (1) and (5)–(7) are

$$2i\left[P_\parallel - q_\parallel \frac{k_x}{\omega} \right] \vec{D} \vec{v}_r = k_x q_\parallel - \omega P_\parallel + P_\perp k_x \vec{v}_r,$$

$$\left[q_\parallel - 3 \frac{k_x}{\omega} k_x \vec{D}_\parallel \right] \vec{D} \vec{v}_r = i \omega q_\parallel - i3k_x k_x \vec{S}^2 \vec{P}_\parallel$$

$$- ik_x \left(4q_\parallel - 3 \frac{k_x}{\omega} k_x \vec{S}^2 \vec{P}_\parallel \right) \vec{v}_r,$$

$$\left[2q_\perp - k_x \frac{k_x}{\omega} \vec{S}^2 (2P_\parallel - P_\perp) \right] \vec{D} \vec{v}_r = i \omega q_\perp - ik_x k_x \vec{S}^2 \vec{P}_\perp$$

$$- ik_x \left(2q_\perp - k_x \frac{k_x}{\omega} \vec{S}^2 \vec{P}_\perp \right) \vec{v}_r,$$  \hspace{1cm} (15)

in which $\vec{S}^2_\perp = P_\perp/\rho_0$ and the perturbed pressure tensor $\vec{P}_\perp = \vec{P}_\perp + i(P_\parallel - P_\perp)(\vec{b}_1 \vec{b} + \vec{b}_1 \vec{b})$ and $\vec{b} = \vec{B}_1/\vec{B}_0 - \vec{b} \vec{B}_1/\vec{B}_0$ are understood. We also assumed $P'_\parallel = 0$ and $P_0' = 0$, and $V_A = \sqrt{\rho_0/\rho_0}$ is the Alfvén speed. The parameter $\chi$ is introduced additionally to Eqs. (14) and (15) to identify the effects of heat conduction by following Ref. 34. It is known that the entropy equations become $d(B^2 P_\parallel/\rho_0)/dt = 0$ and $d(P_\perp/\rho B)/dt = 0$, i.e., the well-known CGL double equations when the heat conduction effect is neglected. When the equilibrium heat flux and heat conducting effect are both ignored by letting $q_\parallel = q_\perp = 0$ and $\chi = 0$, the CGL model is recovered. For $q_\parallel = q_\perp = 0$ and $\chi = 1$, the equilibrium heat flux is supposed to be zero, but the heat conducting effect is still taken into account. Since Eqs. (14) and (15) indicate that the perturbed heat flux still plays a role even if the equilibrium heat flux is disregarded, it is necessary to include the parameter $\chi$. Now, we derive the DR on the basis of Eqs. (12)–(18). From Eq. (18), we have

$$\vec{P}_\parallel = \frac{\omega}{k_x} \rho_0 \vec{v}_r + i \rho_0 \frac{\omega}{k_x} (S_\perp^2 - S_\parallel^2) \vec{D} \vec{v}_r.$$  \hspace{1cm} (19)

Substituting it into Eq. (14) yields

$$\vec{q}_r = -i \frac{1}{\omega} \left[ q_\parallel - 3 \frac{k_x}{\omega} \vec{S}^2 \vec{P}_\parallel \right] \vec{D} \vec{v}_r + k_x \frac{1}{\omega} \left[ 4q_\parallel + \frac{3 k_x}{\omega} k_x \vec{P}_\parallel \left( \frac{\omega^2}{k_x^2} - S_\parallel^2 \right) \right] \vec{v}_r,$$  \hspace{1cm} (20)

Inserting the two formulas above into Eq. (12), we obtain the parallel perturbed velocity in terms of $\vec{v}_r$ as

$$\vec{v}_r = i \frac{\vec{D} \vec{v}_r}{k_x},$$  \hspace{1cm} (21)

where we denote

$$\varphi = -\omega k_x^2 (S_\perp^2 - q_\parallel - 2k_x q_\perp/\rho_0 + 3 \frac{k_x \vec{S}_\perp^2}{\omega} - S_\parallel^2) k_x^4 \omega$$

$$\omega^2 - 3(1 + \chi) \omega k_x^2 S_\perp^2 - 4 \omega k_x q_\parallel/\rho_0 + 3 \frac{k_x \vec{S}_\perp^2}{\omega}$$  \hspace{1cm} (22)

On the other hand, elimination of $\vec{q}_\perp$ from Eqs. (13) and (15) gives rise to

$$\vec{P}_\perp = \frac{k_x}{\omega} \frac{P_\parallel}{\omega^2 - k_x^2 \vec{S}^2_\perp} \vec{D} \vec{v}_r$$

$$+ \frac{2 \omega P_\perp - 2k_x \vec{S}_\perp^2 (2P_\parallel - P_\perp)/\omega}{i(\omega^2 - k_x^2 \vec{S}^2_\perp)} \vec{D} \vec{v}_r.$$  \hspace{1cm} (23)

By inserting $\vec{v}_r$, the formula above is re-expressed in terms of $\vec{v}_r$ as

$$\vec{P}_\perp = i \frac{1}{\omega} \frac{d \Omega}{d \ln \rho} + \frac{2 \Omega \omega}{\omega^2 - k_x^2 \vec{S}^2_\perp} \vec{D} \vec{v}_r,$$  \hspace{1cm} (24)

in which

$$\varphi = \frac{\varphi P_\parallel}{\omega^2 - k_x^2 \vec{S}^2_\perp} + \frac{2 \omega P_\perp - 2k_x \vec{S}_\perp^2 (2P_\parallel - P_\perp)/\omega}{\omega^2 - k_x^2 \vec{S}^2_\perp}.$$  \hspace{1cm} (25)

From Eq. (17), the azimuthal component of the perturbed motion equation, we derive the azimuthal perturbed velocity in terms of $\vec{v}_r$,

$$\vec{v}_0 = -i \frac{1}{\omega} \frac{d \Omega}{d \ln \rho} + \frac{2 \Omega \omega}{\omega^2 - k_x^2 \vec{S}^2_\perp} \vec{D} \vec{v}_r.$$  \hspace{1cm} (26)

Combining Eqs. (16), (24), and (26) to eliminate $\vec{P}_\perp$ and $\vec{v}_0$, we obtain the following mode equation describing the radial perturbed velocity:

$$\left[ \omega^2 - k_x^2 \vec{S}^2_\perp + \frac{\vec{V}_\perp}{\omega} \vec{D} \vec{D} \vec{v}_r + \frac{\omega \theta}{\rho_0} \vec{D} \right] \vec{v}_r$$

$$- \frac{4 \Omega^2 \omega^2}{\omega^2 - k_x^2 \vec{S}^2_\perp} \frac{d \Omega}{d \ln \rho} \vec{D} \vec{v}_r = 0.$$  \hspace{1cm} (27)

Under the local approximation, $kr \gg 1$, all the 1/(kr) terms can be neglected by assuming $\partial_r \vec{D} \vec{v}_r = -k_x^2 \vec{v}_r$, where $k = (k_x^2 + k_x^2)^{1/2}$ is the total wave number and $k_x$ is the radial wave number. By defining $A^2 = S_\parallel^2 + S_\perp^2$, we obtain the DR depicting the MRI in a collisionless and differentially rotating plasma with anisotropic equilibrium heat flux and heat conducting effect.
\[\begin{aligned}
(\omega^2 - k_x A^2) \left( \omega^2 - k_y S^2 \right) G_1 \\
- \left( k_x^2 V_A^2 + \frac{d \Omega^2}{d \ln r} \right) (\omega^2 - k_y A^2) (\omega^2 - k_x S^2) G_1 \\
+ k_y S \left[ (\omega^2 - k_x S^2) G_2 - 2G_1 - 2k_x k_y S G_2 + 2\omega k_x G_2 \frac{\partial q_\perp}{P_\perp} \right]
\times (\omega^2 - k_y A^2) - 4\Omega^2 \omega^2 (\omega^2 - k_x S^2) G_1 = 0,
\end{aligned}\]

in which

\[G_1 = \omega^4 - 3(1 + z) k_x^2 S^2 (\omega^2 - 4\omega k_x S^2 q_\parallel / P_\parallel + 3k_y^2 S^4),\]

\[G_2 = -\omega^2 k_x^2 (2S^2 - S_\perp^2) - 2\omega k_x k_y S q_\parallel / P_\parallel + 3k_y^2 S (2S^2 - S_\perp^2).\]

**B. Discussion**

As a high-order equation for \(\omega\), the dispersion (28) needs to be simplified for further discussion. Letting \(k_x = 0\), and then the dispersion shows the following three modes:

\[\begin{aligned}
\left\{ \begin{array}{l}
\omega^2 = 2k_y S^2, \\
G_1 = 0, \\
(\omega^2 - k_x^2 A^2)^2 - \frac{d \Omega^2}{d \ln r} (\omega^2 - k_y^2 A^2) - 4\Omega^2 \omega^2 = 0.
\end{array} \right. \tag{29}
\]

The former two modes are generated by algebraic manipulation when deriving the DR and need to be checked whether they are extraneous roots or not. For \(\omega^2 = 2k_y S^2\), since \(k_x = 0\), there is \(\frac{d}{d \ln r} \tilde{v} = 0\) and \(\frac{\partial}{\partial \ln r} P_\perp = 0\), and it is found that \(\tilde{q}_\parallel = P_\parallel = \tilde{v} = 0\), and then the set of equations is reduced to

\[\begin{aligned}
&k_x \tilde{q}_\perp - \omega \tilde{P}_\perp = 0, \\
&\omega \tilde{q}_\perp - k_y S^2 \tilde{P}_\perp = 0.
\end{aligned}\]

And

\[\begin{aligned}
&-i(\omega^2 - k_x^2 A^2) \tilde{v}_\parallel - 2\omega \omega \tilde{v}_\parallel = 0, \\
&-i\omega(\omega^2 - k_y^2 A^2) \tilde{v}_\parallel + \frac{d \Omega^2}{d \ln r} (\omega^2 - k_y^2 A^2) \tilde{v}_\parallel + 2\omega \omega \tilde{v}_\parallel = 0.
\end{aligned}\]

The former two equations yield the first mode and the latter two equations yield the third mode. Specifically, when \(\tilde{v}_\parallel = \tilde{v}_\parallel = 0\), Eq. (30a) gives \(\tilde{P}_\perp = \text{Arbitrary}\) and \(\tilde{q}_\perp / k_x\). That is, only the two perturbations exist and other perturbations are null, meaning that dominated by the parallel acoustic wave mode, only the perpendicular pressure and heat flux can generate fluctuations. Since the fluid density and velocity are kept unperturbed, the perturbations of pressure and heat flux are induced by the perpendicular thermal temperature \(T_\perp\). Concerning the MRI mode described by Eq. (30b), \(\tilde{q}_\perp = \tilde{P}_\perp = 0\), \(\tilde{v}_\parallel = \text{Arbitrary}\), and \(\tilde{v}_\parallel \) is determined by Eq. (26). Hence, there exist only radial and azimuthal fluid velocity fluctuations. It implies that the MRI can be generated solely by fluid fluctuation not essentially by heat flux or the thermal pressure but the differential rotational effect, the so-called Velikhov effect.\(^1\)

Now, we look into the second mode \(G_1 = 0\). One finds \(\tilde{v}_\parallel = \tilde{v}_\parallel = 0\), \(\tilde{v}_\parallel = \text{Arbitrary}\), and

\[\begin{aligned}
\tilde{q}_\perp = \frac{2\omega k_y q_\parallel}{\omega^2 - k_x^2 S^2} \tilde{v}_\parallel, \quad \tilde{P}_\parallel = \frac{\omega k_x}{k_x} \rho \tilde{v}_\parallel, \\
\tilde{q}_\parallel = \frac{k_c}{\omega} \left[ 4q_\parallel + 3k_c / \omega P_\parallel (\omega^2 - S^2) \right] \tilde{v}_\parallel, \\
\tilde{P}_\parallel = \frac{k_c}{\omega} P_\parallel + \frac{2k_c q_\parallel}{\omega^2 - k_x^2 S^2} \tilde{v}_\parallel.
\end{aligned}\]

According to the expression of \(G_1\), this mode does not contain any rotation effect. It is driven purely by fluid pressure and heat flux and is similar to the results reported in Ref. 34. Hence, we do not discuss the two modes in detail here but rather focus on the third MRI mode. The dispersion can be rewritten as

\[\begin{aligned}
\frac{d \Omega^2}{d \ln r} + k_x^2 V_A^2 + k_y^2 A^2 \left( \frac{d \Omega^2}{d \ln r} + k_x A^2 \right) = 0,
\end{aligned}\]

where \(k_x^2 = 4\Omega^2 + d\Omega^2 / d \ln r\) is the squared epicyclic frequency. This mode includes the pressure effect but not the heat flux vector or the heat conducting effect. The instability criteria are altered by the uncertainty of the sign of \(A^2\). Defining \(C_1 = 2k_x A^2 + k_y^2 A^2 (d \Omega^2 / d \ln r + k_x A^2)\), and the discriminant \(\Delta = C_1^2 - 4C_2\), one finds that:

1. For \(A^2 > 0\), i.e., \(T_\perp > T_\parallel\) in the weak field limit, the discriminant of the formula above is always positive and hence, the instability takes place when \(C_2 < 0\), namely

\[\frac{d \Omega^2}{d \ln r} + k_x^2 V_A^2 + k_y^2 (S_\perp^2 - S_\parallel^2) < 0.\]

2. When \(A^2 < 0\) and the discriminant is still positive, which requires \(d \Omega^2 / d \ln r > 4\Omega^2 \sqrt{-k_x^2 A^2 - 4\Omega^2} \) or \(d \Omega^2 / d \ln r < -4\Omega^2 \sqrt{-k_x^2 A^2 - 4\Omega^2}\), the instability occurs when \(C_2 < 0\) or \(C_1 < 0\), which yields \(d \Omega^2 / d \ln r > -k_x^2 A^2\) or \(d \Omega^2 / d \ln r < -2k_x^2 A^2 - 4\Omega^2\), respectively. Note that \(-2k_x^2 A^2 - 4\Omega^2 > -4\Omega^2 \sqrt{-k_x^2 A^2 - 4\Omega^2}\) and \(-k_x^2 A^2 > 4\Omega \sqrt{-k_x^2 A^2 - 4\Omega^2}\), and thus, the critical condition responsible for instability is \(d \Omega^2 / d \ln r > -k_x^2 A^2\) or \(d \Omega^2 / d \ln r < -4\Omega^2 \sqrt{-k_x^2 A^2 - 4\Omega^2}\).

3. When \(A^2 < 0\) and the discriminant is negative, \(\omega\) is not a pure imaginary number anymore, and overinstability comes into being with \(\omega = \omega_0 + i \gamma\), where \(\omega_0\) and \(\gamma\) are both real numbers. \(\Delta < 0\) requires \(-4\Omega^2 \sqrt{-k_x^2 A^2 - 4\Omega^2} < d \Omega^2 / d \ln r < 4\Omega^2 \sqrt{-k_x^2 A^2 - 4\Omega^2}\) with a growth rate \(\gamma = (2\sqrt{C_2 - C_1})^{1/2}\).

Combining the last two cases together, we obtain the boundary conditions accounting for the instability or overinstability for \(A^2 < 0\) as follows:

\[\frac{d \Omega^2}{d \ln r} > -k_x^2 A^2,\]

and
The two formulas above and Eq. (33) are the critical conditions for the MRI to come into being in a collisionless magnetized plasma. The heat flux vector and heat conduction have no effects on the MRI in the absence of perpendicular wave number. The anisotropic pressures show significant effects on the instability criteria. Compared to the classical criterion $d\Omega^2/d \ln r + k^2 V_A^2 < 0$, the pressure not only modifies the classical criterion [see Eq. (33)] but also introduces new unstable regions [see criteria (34) and (35)]. It appears that the criterion (33) can be reduced to a classical one in the collisional plasma by letting $S_\perp = S_{||}$. However, this manipulation is not allowed since we have adopted Eqs. (4) and (5). When the heat conduction effect is neglected, we go to the CGL mode in which the parallel and perpendicular pressures are treated differently and satisfy

$$ P_\parallel B^2 / \rho^3 = \text{constant}, $$

$$ P_\perp / (\rho B) = \text{constant}, $$

respectively. In other words, one has $S_\perp \neq S_{||}$ in the collisionless plasma. However, when dealing with the MRI in a collisionless plasma, Refs. 27 and 29 assume $P_\perp = P_{||} = P_0$, but the perturbed pressures $P_\perp$ and $P_\parallel$ are determined by the drift kinetic equations. In this case, $A^2 = V_A^2$, the DR (32) becomes the classical one. In other anisotropic environments, for instance, in the quiet solar wind, $P_\parallel / P_\perp$ is about 2 (Ref. 31), and so $S_{||}^2 = 2S_\perp^2$ and $A^2 \approx -S_\perp^2 < 0$ in the weak field limit. The instability criteria become

$$ d\Omega^2 d \ln r - k^2 S_\perp > 0 $$

and

$$ d\Omega^2 d \ln r < 4\Omega(k_2 S_{||} - \Omega). $$

Equation (37) implies that the pressure plays a stabilizing effect on the instability. According to the criterion (33), if $T_\perp > T_{||}$, the pressure also shows a stabilizing effect on the instability. Equation (38) indicates, however, a destabilizing effect on the instability. Hence, it is not certain that the pressures have a destabilizing or a stabilizing effect, which depends on the competition between the perpendicular pressure, parallel pressure, the local Alfvén velocity, as well as the parallel wave number (recall that we let $k_z = 0$ above). A limiting case is in which $[k_x A_x] \gg \Omega$. The criteria (33) and (34) are not satisfied, where we assume that $d\Omega/d \ln r$ is on the order of $\Omega$. Criterion (35) is simplified to $\kappa^2/(4\Omega) < -k^2 A^2$. Since the right-hand side of this inequality is much greater than the left-hand side, the inequality is always satisfied. That is, for $A^2 \gg \Omega/k_z$, there must be instability. Hence, the perturbation is stable if $T_\perp > T_{||}$ in this case.

Not only the instability criteria but also the growth rate is modified by the pressures. In a Keplerian disk, $d\Omega^2/d \ln r = -3\Omega^2$, and in the classical MHD mode, the fastest grow-

\[ \frac{d\Omega^2}{d \ln r} < 4\Omega \sqrt{-k^2 A^2 - 4\Omega^2}. \] (35)

ing mode appears when $k V_A = (15/16)^{1/2} \Omega$ with the growth rate $\gamma = 3\Omega/4$. When $A^2 > 0$, the dispersion (32) yields the same results, namely the fastest mode occurring at $k^2 A^2 = 15\Omega^2/16$. Since $A^2$ can be negative, a new fastest mode comes into being. Recall that for $k^2 A^2 < -\Omega^2/16$, the growth rate is $\gamma = (2\sqrt{2}/C_0 - C_1)^{1/2}/2$, and in this case, the growth rate increases monotonically as $-k^2 A^2$ increases. For example, when $k^2 A^2 = -\Omega^2$, one finds $\gamma = (5/4)^{1/2} \Omega$. In an isotropic collisional plasma, the pressure effect on the MRI may be different and it will be discussed in Sec. IV.

When $k_z = 0$, $G_1 = \omega^2$ and $G_2 = 0$, and the dispersion (28) goes to

\[ \omega^2 = \kappa^2 + k^2 V_A^2 + 2k^2 S_\perp^2. \] (39)

Since the last two terms on the right-hand side of this formula are always positive, only the first term can drive an instability. In the Keplerian differential rotation system, $\kappa^2 = \Omega^2$ and as a result, the system is always stable.

### IV. ISOTROPIC PLASMAS WITH SELF-GRAVITATION

In a collisional plasma, the kinetic thermal pressure is described by a scalar quantity $P$, and the state equation in an ideal plasma is well-known as

\[ \frac{d}{dt}(P \rho^{-\Gamma}) = 0, \] (40)

in which $\Gamma$ is the adiabatic index. Replacing Eqs. (4)–(7) by the state equation above and assuming $P = P I$ in the motion Equation (1), we obtain the following DR after some algebraic manipulation,

\[ \frac{-i\omega [\omega^2 - k^2 c_s^2] + \frac{1}{\omega^2} \left( \frac{d\Omega^2}{d \ln r} + k^2 V_A^2 + 4\Omega^2 \omega^2 \right)}{\omega^2} \]

\[ + \frac{i\omega \omega^2}{\omega^2 - k^2 c_s^2} \left[ \frac{P_{\parallel} k_z^2}{\rho_0} \left( 1 - \frac{\Gamma \omega^2}{\omega_s^2 - k^2 c_s^2} + \frac{\rho_0 (k_z c_s^2)^2}{\rho_0 (\omega_s^2 - k^2 c_s^2)} - \omega^2 k^2 \right) \right] \]

\[ \times \frac{1}{k^2 \omega_s^2 - k^2 c_s^2 + \omega^2 k_z^2 / k^2 \rho_0 ^4} \left[ k - \frac{\rho_0 (\omega_s^2 - k^2 c_s^2)}{\rho_0} \right] \]

\[ = \frac{k_z^2 / \omega^2}{\omega^2 - k^2 c_s^2} \left[ \frac{P_0}{\rho_0} \left( \Gamma - 1 \right) \omega^2 + k^2 c_s^2 \right] \]

\[ \frac{-i P_{\parallel} k_z^2}{\omega^2} \left[ \frac{P_0}{\rho_0} \frac{1}{\omega^2 (\omega_s^2 - k^2 c_s^2)} \right] \]

\[ + \frac{i\omega \omega^2}{\omega^2 - k^2 c_s^2} \left[ \frac{P_{\parallel} k_z^2}{\rho_0} \frac{\Gamma \omega^2}{\omega_s^2 - k^2 c_s^2} + \frac{P_0}{\rho_0} \frac{\omega_s^2}{\omega_s^2 - k^2 c_s^2} \right], \] (41)

in which $\omega_s = (4\pi G \rho_0)^{1/2}$ is the local plasma Jeans frequency and $c_s = (\Gamma P_0/\rho_0)^{1/2}$ is the sound speed. The dispersion describes the differential rotational instability with self-gravitational effect in the collisionless plasma under the local Wentzel-Kramers-Brillouin (WKB) approximation, which requires $kL \gg 1$, where $L$ is the local scale length of the equilibrium profiles.

#### A. Incompressible modes

In the incompressible limit, i.e., $c_s \to \infty$, the dispersion (41) becomes
\[ 1 - \frac{k^2}{k^2} \left( \frac{dQ^2}{dl} + k^2 V_A^2 + \frac{4Q^2 \omega^2}{\omega^2 - k^2 V_A^2} \right) = \frac{1}{k^2} \left( P_0 \rho_0 - \frac{\omega^2}{k^2} \frac{k^2}{\rho_0} \right) + \frac{\omega^2}{k^2} \frac{k^2}{\rho_0} \frac{\rho_0}{\rho_0} \frac{\rho_0}{\rho_0}. \quad (42) \]

Under the local WKB approximation, the last term is much smaller than unity and can be neglected since it is proportional to \(1/kL\), where \(L = \rho_0/\rho_0^2\) is defined as the scale length of the density gradient. By introducing a parameter \(\delta \ll 1\) and defining \(kL = \delta^{-1}\), all the terms \(O(\delta)\) should be disregarded. The second term on the right-hand side of the equation above seems to be proportional to \(\delta^2\) and should be ignored, but in fact, this term should be retained if \(\omega_0/\omega \geq \delta^{-1}\). Here, we do not consider the special case of \(k_0 \ll k\).

We investigate the case of \(\omega_0 = 0\) first. By using the equilibrium condition \(8\), the DR \(42\) is reduced to

\[ \omega^2 - k^2 V_A^2 - \frac{k^2}{k^2} \left( \frac{dQ^2}{dl} + (g - \Omega^2) \frac{d\ln \rho_0}{dr} \right) = 0, \quad (43) \]

which is identical to Mikhailovskiǐ’s result.\(^{39}\) Ignoring the density gradient effect, we arrive at the classical DR of MRI (Ref. 7)

\[ \omega^2 - k^2 V_A^2 - \frac{k^2}{k^2} \left( \frac{dQ^2}{dl} + \frac{4Q^2 \omega^2}{\omega^2 - k^2 V_A^2} \right) = 0. \quad (44) \]

From the equation above, we see that in the general case, one has \(\omega \sim O(\Omega, kV_A)\). Hence, when \(\omega_0 \geq \Omega \delta^{-1}\), the local self-gravitational effect cannot be neglected. A typical protostellar disk consists of molecular gas with a temperature ranging from 10 K in its outer regions to 10^3 K near the central star and may spread over 100 AU (Ref. 6). We assume a 1\(M_\odot\) central protostar surrounded by a disk with a mass much less than 1\(M_\odot\). For \(r = 10\) AU, a characteristic mass density can be \(5 \times 10^{-7}\) kg m\(^{-3}\). By virtue of \(G = 6.67 \times 10^{-11}\) m\(^3\) kg\(^{-1}\) s\(^{-2}\), \(M_\odot = 1.9891 \times 10^{30}\) kg, and \(1\) AU = \(1.50 \times 10^{11}\) m, one has \(\Omega_0 \approx 2.05 \times 10^{-8}\) s\(^{-1}\), \(\Omega \approx 6.27 \times 10^{-8}\) s\(^{-1}\), and \(\omega_0^2/\Omega^2 \approx 11\). On the other hand, taking \(\Gamma = 5/3\), plasma mass \(m = 1.67 \times 10^{-27}\) kg, and \(T = 50\) K, the sound speed is about \(c_s \approx 6.9 \times 10^3\) m s\(^{-1}\). Hence, for \(k \gg 10^{-11}\) m\(^{-1}\), the fluid can be considered as incompressible. In other words, only for the perturbations with wavelengths which are comparable to or greater than 1 AU, the sound speed is not ignorable compared to the Jeans frequency or the angular velocity. Now, we take into account the self-gravitational effect of the plasma mass density perturbation, and then the dispersion becomes

\[ \omega^4 - \eta_1 \omega^2 + \frac{k^4 V_A^2}{k^2} \eta_0 = 0, \quad (45) \]

where

\[ \eta_1 = 2k^2 V_A^2 + \frac{k^2}{k^2} \left( \kappa^2 - \frac{\omega^2}{\rho_0} \rho_0 - \omega_0^2 \delta^2 \right) \quad (46) \]

and

\[ \eta_0 = k^2 V_A^2 + \frac{dQ^2}{dl} - g_e \frac{\rho_0}{\rho_0} - \omega_0^2 \delta^2 \quad (47) \]

The instability takes place when \(\eta_0 < 0\), that is,

\[ \frac{dQ^2}{dl} - g_e \frac{\rho_0}{\rho_0} - \omega_0^2 \delta^2 < 0 \quad (48) \]

in the weak field limit. Here, we define the local effective gravitational acceleration \(g_e = g - \Omega^2\). In the criterion above, the first term represents the Velikov effect, which drives an instability when \(dQ/\Omega d < 0\). The second term represents the local gravity interchange mode. It is well known that the interchange mode is unstable when the density gradient is in the opposite direction of the gravity, that is, \(g_e \rho_0 < 0\). The criterion above confirms this point. The last term is due to the self-gravitational effect, which is always destabilizing. In a Keplerian system of \(\rho_0 = 0\), \(g_e = 0\), the criterion above becomes \(- \Omega^2 - \omega_0^2 \delta^2 < 0\). The system is always magnetohydrodynamically unstable. For uniform rotation \(dQ/\Omega d < 0\) in the homogenous plasma, the Jeans term solely drives the instability.

**B. Non-rotational mode in compressible plasmas**

In a homogenous plasma with \(P_0 = 0\), \(\rho_0 = 0\), and \(\Omega = 0\), i.e., in the absence of rotation, the equilibrium condition \(8\) requires \(g = 0\). That is, the gravity in the unperturbed state is ignored. Strictly speaking, the equilibrium cannot be regarded as homogenous for a typical Jeans problem. However, the famous Jeans swindle\(^{40,41}\) enables one to study local perturbations in a system which is assumed to be homogenous on the wavelength scale. The DR \(41\) now is reduced to

\[ \frac{\omega^2 - k^2 c_s^2}{\omega^2} + \frac{k^2 V_A^2}{\omega^2} = \frac{k^2(\omega^2 - k^2 c_s^2)(\omega^2 - k^2 c_s^2 + \omega_0^2 k^2/k^2)}{k^2(\omega^2 - k^2 c_s^2)(\omega^2 - k^2 c_s^2 + \omega_0^2 k^2/k^2)} = 0. \quad (49) \]

By defining \(V_A^2 = c_s^2 - \omega_0^2 k^2/k^2\), the dispersion above is simplified to

\[ \omega^4 - \omega^2 (k^2 V_A^2 + 2k^2 V_A) + k^2 V_A^2 k^2 V_A^2 = 0, \quad (50) \]

which is identical to the previous result for typical Jeans instability (see, for example, Ref. 42). The Jeans instability occurs provided that \(k^2 V_A^2 V_A^2 < 0\). For non-zero magnetic field and \(k_0 \neq 0\), the criterion above reduces to \(V_A^2 < 0\), the same as the one in the non-magnetized case, viz., the classical criterion for Jeans instability

\[ \omega_0^2 > k^2 c_s^2. \quad (51) \]

For \(k = 0\), the critical condition for the Jeans instability to occur becomes \(\omega_0^2 > k^2 (c_s^2 + V_A^2)\). The Alfvén frequency contributes to the criterion and shows stabilization effect on the instability.
C. Rotational mode in compressible plasmas

For $P' = 0$, $\rho' = 0$, and $\Omega \neq 0$, the dispersion (41) is reduced to

$$\frac{\omega^2 - k^2 V^2}{\omega^2 - k^2 c_s^2} - \frac{1}{\omega^2} \left( \frac{d\Omega^2}{dl} + k^2 V_A^2 + \frac{4\Omega^2 \omega^2}{\omega^2 - k^2 c_s^2} \right) = 0. \quad (52)$$

A special case is $k_z = 0$. It is easy to see that the pressure terms are not contained in the dispersion, and the instability criterion becomes $d\Omega^2/d\ln r + k^2 V_A^2 < 0$, which can be recovered by assuming $S_z = S_r$ in the criterion (33) for the collisionless plasma. However, as aforementioned, this smooth transition from a collisionless plasma to a collisional one is not allowed. Hence, in the collisionless plasma in the case of $k_z = 0$, compressibility effect still affects the instability criterion. Another limiting case is $k_z = 0$ and then the DR becomes

$$\omega^2 = k^2 + k^2 c_s^2 + k^2 V^2. \quad (53)$$

It’s easy to see that the instability takes place when

$$\omega^2 > k^2 c_s^2 + \Omega^2 + k^2 V_A^2, \quad (54)$$

in a Keplerian rotation system. The local self-gravitational effect overcomes the destabilizing effect due to not only the pressure but also the local Alfvén waves as well as the centrifugal force. In a sufficiently cold and dense plasma with low rotation frequency, this condition can be satisfied. Comparing Eq. (53) with Eq. (39), we see that the pressure effect is represented by the $2S_z^2$ term in an anisotropic plasma whereas the term of $c_s^2$, i.e., $5P_0/\rho_0$ for an isotropic plasma. The coefficients are different even if one supposes that $P_\perp$ is equal to $P_0$. Recall that $\Gamma$ is defined as $(2 + D)/D$, where $D$ is the dimension. Hence, the anisotropic case, or more precisely, the cross-field motion, is two-dimensional and corresponds to $\Gamma = 2$ (Ref. 27). Meanwhile, the parallel (with respect to the magnetic field) motion is one-dimensional and corresponds to $\Gamma = 3$, namely, $3k_{\parallel}S_\parallel$, which can be seen from the expression of $G_1$. The instability growth rate $\gamma = -i\omega$ is

$$\gamma^2 = k^2 \omega^2 - \Omega^2 - k^2 c_s^2 + V_A^2. \quad (55)$$

Now, we examine the effects of compressibility and self-gravitation on the MRI directly on the basis of the DR (52) in order to obtain the instability criterion. In terms of dimensionless variables

$$\tilde{\gamma} = (-i\omega/\omega_A)^2, \quad \Lambda = V^2/V_A^2, \quad k^2/k_s^2 = 1 + \chi^2, \quad R_m = \Omega/\omega_A, \quad \zeta = k^2/(2\Omega^2). \quad (56)$$

With $\omega_A = k_c V_A$, the DR can also be written as

$$\tilde{\gamma}^3 + \tilde{\gamma}^2 \Theta_2 + \tilde{\gamma} \Theta_1 + \Theta_0 = 0, \quad (57)$$

in which

$$\Theta_2 = (1 + \chi^2)\Lambda + 2 + \chi^2 + 2R_m^2r, \quad \Theta_1 = (1 + \chi^2)(1 + 2\Lambda) + 2R_m^2\Lambda + 2R_m^2(\zeta - 2), \quad (58)$$

$$\Theta_0 = \Lambda(1 + \chi^2) + 2R_m^2(\zeta - 2)\Lambda.$$
Considering that $\tilde{\gamma}^2 (2\tilde{\gamma}_1 + \Theta_2) > 0$ when $\zeta < \zeta_1$, the instability criterion is simplified to $G_0 < 0$, namely,

$$2R_m (\zeta - 2) + 1 + \chi^2 < 0. \quad (67)$$

That is the classical Balbus’ criterion for the occurrence of MRI in the incompressible ideal plasmas, which also can be directly derived from the DR (52) by letting $\omega^2 \ll k^2 V^2$. In conclusion, Eq. (66) displays the instability criterion of differential rotation in the presence of self-gravitation effect in the compressible ideal plasmas. However, their presence has no effect on the critical condition of instability if the local Jeans frequency is less than the acoustic frequency, whereas the instability growth rate is altered by them.

Using the definition of $G_0$, we find

$$\begin{align*}
\Theta_2 &= G_0 + \tilde{t}_2, \\
\Theta_1 &= (1 + \Lambda) (G_0 + \tilde{t}_1), \\
\Theta_0 &= \Lambda G_0,
\end{align*}$$

with $\tilde{t}_1 = \frac{\Lambda V}{c_m} (4R_m^2 + 1 + \chi^2)$ and $\tilde{t}_2 = 4R_m^2 + 1 + (1 + \chi^2)\Lambda$. For overinstability, $\tilde{\gamma} = \tilde{\gamma}_0 + i\tilde{\gamma}_1$ and $\omega = \omega_0 + i\omega_r$, where $\tilde{\gamma}_0, \tilde{\gamma}_1, \omega_0, \omega_r$, and $\gamma$ are all real numbers. The growth rate is determined by $\gamma^2 = (\tilde{\gamma} - \tilde{\gamma}_0)/2$ and the oscillation part is $\omega^2 = (\tilde{\gamma} - \tilde{\gamma}_0)/2$. Meanwhile, the imaginary part of $\gamma$ is governed by $\tilde{\gamma}_1 = -2\omega_0\gamma$, implying that $\tilde{\gamma}_1$ should be negative for an instability. The real and imaginary parts of Eq. (57) are

$$3\tilde{\gamma}_0^3 - 3\tilde{\gamma}_0^2\tilde{\gamma}_1 + (\tilde{\gamma}_0 - \tilde{\gamma}_1^2)\Theta_2 + \tilde{\gamma}_0\Theta_1 + \Lambda G_0 = 0 \quad (69)$$

and

$$\tilde{\gamma}_1 (3\tilde{\gamma}_0^2 - \tilde{\gamma}_1^2 + 2\tilde{\gamma}_0\Theta_2 + \Theta_1) = 0. \quad (70)$$

The formula above yields

$$\tilde{\gamma}_1^2 = 3\tilde{\gamma}_0^2 + 2\tilde{\gamma}_0\Theta_2 + \Theta_1, \quad (71)$$

which is inserted into the real part giving rise to

$$3\tilde{\gamma}_0^3 + \tilde{\gamma}_0^2\Theta_2 + \frac{1}{4}\tilde{\gamma}_0 (\Theta_1 + \Theta_2^2) - \frac{1}{8}(\Lambda G_0 - \Theta_1\Theta_2) = 0, \quad (72)$$

defining the left-hand side of the equation above as a function of $\tilde{\gamma}_0$, $H(\tilde{\gamma}_0)$. The intersection point between the curve $H(\tilde{\gamma}_0)$ and $H = 0$ is the root of the equation above. $H$ is a cubic equation about $\tilde{\gamma}_0$ and thus has two flex points determined by $\partial^2 H / \partial \tilde{\gamma}_0^2 = 0$, which is

$$3\tilde{\gamma}_0^2 + 2\tilde{\gamma}_0\Theta_2 + (\Theta_1 + \Theta_2^2)/4 = 0, \quad (73)$$

giving the value of points as

$$\tilde{\gamma}_{1,2} = -\Theta_2 \pm \frac{1}{2}\sqrt{-3\Theta_1}. \quad (74)$$

It has been shown that $\Theta_2^3 - 3\Theta_1 > 0$ is always satisfied and so the two points both belong to the real numbers field. Since the occurrence of instability does not depend on the sign of $\tilde{\gamma}_0$, the analysis is largely simplified. The only constraint on $\tilde{\gamma}_0$ is presented in Eq. (71), which should be positive. Note that this formula is identical to Eq. (59) and they have the same solutions. Hence, the roots of Eq. (72) should be in $(-\infty, \tilde{\gamma}_-) \cup (\tilde{\gamma}_+, +\infty)$ to ensure that $\tilde{\gamma}_1^2 > 0$. Considering that $\tilde{\gamma}_- < \tilde{\gamma}_1 < \tilde{\gamma}_+$ and noting that $H(\tilde{\gamma}_1) = H(\tilde{\gamma}_-)$, the critical condition responsible for instability is simplified to $H(\tilde{\gamma}_-) > 0$ or $H(\tilde{\gamma}_+) < 0$, or in the form

$$\begin{align*}
\Lambda G_0 &< -\frac{2}{27} (\{(G_0 + \tilde{t}_2)^2 - 3(1 + \Lambda)(G_0 + \tilde{t}_1)\})^{3/2} \\
&- \frac{2}{27} (G_0 + \tilde{t}_2)^3 + \frac{1}{3} (1 + \Lambda)(G_0 + \tilde{t}_2)(G_0 + \tilde{t}_1) \quad (75)
\end{align*}$$

or

$$\begin{align*}
\Lambda G_0 &> \frac{2}{27} (\{(G_0 + \tilde{t}_2)^2 - 3(1 + \Lambda)(G_0 + \tilde{t}_1)\})^{3/2} \\
&- \frac{2}{27} (G_0 + \tilde{t}_2)^3 + \frac{1}{3} (1 + \Lambda)(G_0 + \tilde{t}_2)(G_0 + \tilde{t}_1). \quad (76)
\end{align*}$$

As a result, we obtain the full unstable boundary condition. The values of $t_1$ and $t_2$ are fixed when the other parameters $\Lambda$, $R_m$, and $\chi^2$ are given, and then the two criteria above are inequalities with respect to $G_0$. According to the expression of $G_0$, $G_0 \geq G_0(\zeta)$. Owing to the uncertainty of the value and sign of $\zeta$, criteria (75) and (76) can be satisfied. That is, the first step is to check whether the criterion (67) is satisfied or not. If $G_0 < 0$, there exists pure instabilities, and if $G_0 > 0$, pure instabilities do not come into being. However, whenever $G_0$ is positive or negative, overinstabilities can occur. Since in the case of $G_0 < 0$, we are sure that the perturbation is unstable, and there is no need to consider whether an overinstability can occur or not. Hence, only for $G_0 > 0$, instability criteria (75) and (76) are needed. When either one is satisfied, there will be an overinstability with a growth rate $\gamma > \omega_0$.

V. CONCLUSION

We use the modified CGL mode containing anisotropic heat flux vector to investigate the differential rotational instability in a collisionless magnetized plasma. The analytic dispersion relation is derived and presented in Eq. (28). Limiting cases are specifically discussed by assuming that only the longitudinal wave number or transverse wave number is present. The dispersion (32) describes the pressure effect on the MRI in a collisionless magnetized plasma, whereas the heat flux vector shows no effect on the instability in the absence of cross-field wave numbers. The existence of anisotropic pressure is shown to introduce new unstable regimes expressed in Eqs. (34) and (35) when $A^2 < 0$, viz., the perpendicular thermal temperature is less than the parallel temperature in the weak field limit. The instability criterion for positive $A^2$ is presented in Eq. (33).

Section IV discusses the MRI in a collisional magnetized plasma described by a scalar pressure without a heat flux by taking into account self-gravitational effect when deriving the dispersion relation (41). According to this dispersion, we discuss the incompressible mode, non-rotational mode in compressible plasmas, and differential rotational mode in compressible plasmas. In the incompressible case,
the instability criterion (48) indicates that the Jeans term associated with the density gradient plays a destabilizing effect on the MRI. We especially emphasize the discussion on the last mode. According to the dispersion relation (52), we derive the critical condition responsible for a pure instability presented in Eq. (67), which is identical to the classical one obtained in the incompressible ideal plasmas. The scalar pressure does not play a role in the instability criterion in this case. From the classical dispersion (44), we see that there are no overinstabilities in the incompressible limit, whereas in the compressible plasmas, there may be overinstabilities. In this case, the instability takes place provided that either criterion (75) or (76) is satisfied. The scalar pressure plays an effect on both the instability criteria and growth rate.

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