A novel approach to modelling time-between failures of a repairable system

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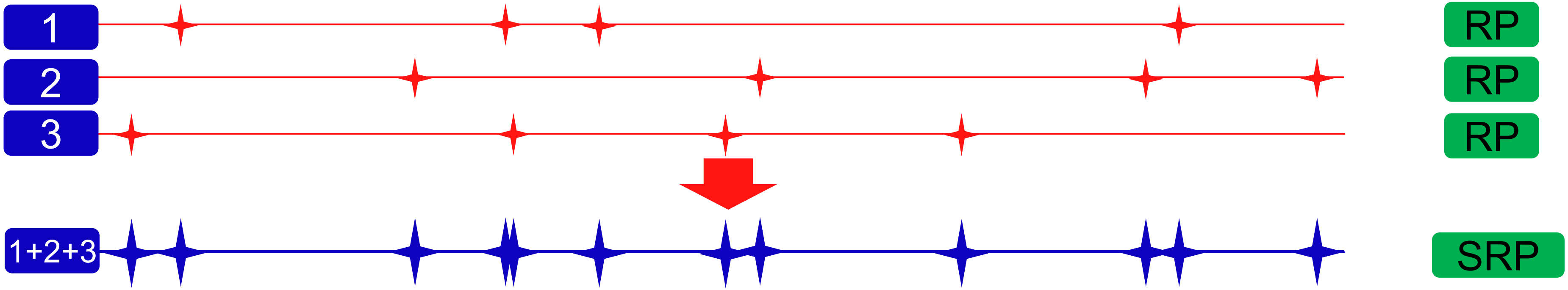
Wu, S., 2018. A failure process model with the exponential smoothing of intensity functions. European Journal of Operational Research. https://doi.org/10.1016/j.ejor.2018.11.045

The failure process of a multiple component system

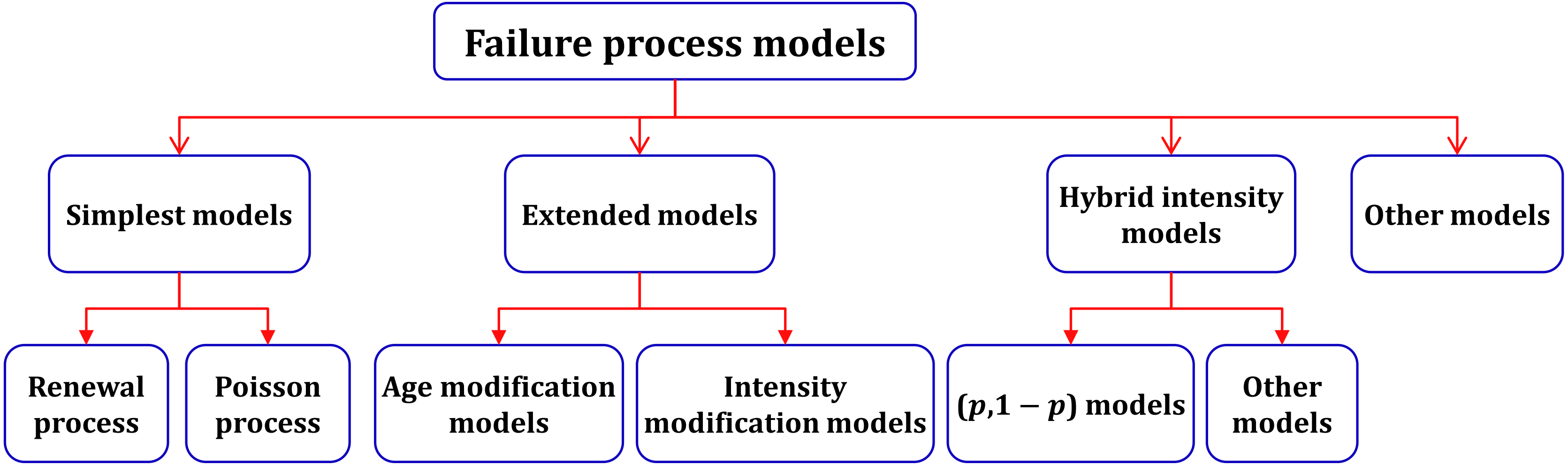
- Consider a series system composed of three components

- Assumptions
  - Once a component fails, it is immediately replaced with an identical one,
  - The failure mode is not recorded/unknown
- The failure process of the system is a super-imposed renewal process (SRP).



Literature review

- The GP has been extensively studied since its introduction in 1988, mainly due to its elegance and convenience in deriving mathematical properties in applications



Limitations in the existing models

- Limitations of the practical application of SRP (super-imposed renewal process):
  - In reality, the number of failures of a system is limited
  - Too many parameters in the SRP

  ➔uncertainty of the parameters in a model becomes large

- A limitation in the existing models: Only considering the same effectiveness of maintenance upon failures in the lifecycle of a system
  - Replacing different components causes different effectiveness of maintenance

- A possible solution
  - Constraint: unknown failure modes + limited number of failures
  - Solution: use the Exponential Smoothing methods in failure intensity function

Two forecasting models

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- Model 1: Simple exponential smoothing method

$$\hat{x}_{n+1} = \alpha x_n + (1-\alpha)\hat{x}_n$$

  - where: $\hat{x}_n$ is the last period's forecast; x_n is the last periods actual value; and $0 < \alpha < 1$
  - We modify the simple exponential smoothing method and think of the following method

$$\hat{x}_{n+1} = \frac{1}{p}\sum_{i=0}^{p-1} \alpha^i x_{n-i}$$

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- Understanding the time series from a failure process perspective
  - x_n is the time to the nth failure, i.e., it is the time to the latest failure, i.e., "*youngest failure*" ..., x_{n-p+1} is the time to the $(n-p+1)$th failure, i.e., "*older failure*"
  - $\alpha^i x_{n-i}$ implies that more weight on the oldest failure

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- We consider two models
  - **The exponential smoothing of intensity model (ESI)**
  - **The simply moving average of intensity model (SMI)**

The failure process of a multiple component system

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  - At the 3rd failure, we assume component 3 fails

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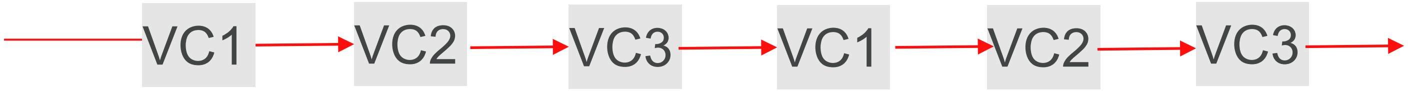
  - At the 1st failure, we assume one of the components, component 1 say, fails;
  - At the 2nd failure, we assume another component, component 2 say, fails
  - At the 3rd failure, we assume component 3 fails
  - At the 4th failure, we assume component 1 fails, …



The exponential smoothing of intensity model

- Assume a real-world system is composed of ***m*** components. The reliability of each component is unknown.

- We approximate the failure process of the above system with that of the following virtual system.

- Assume a system composed of ***m*** components with failure rate functions $\frac{1}{m}\rho^{m-1}\lambda_0(t), \frac{1}{m}\rho^{m-2}\lambda_0(t), \dots, \frac{1}{m}\lambda_0(t)$, respectively. That is, before the 1st failure of the system, the failure intensity function of the system is assumed to be

$$\lambda(t|\mathcal{H}_{t-}) = \frac{1}{m}\sum_{k=0}^{m-1}\rho^{m-k-1}\lambda_0(t)$$

The exponential smoothing of intensity model

- Then for $1 \leq N_t < m$, we have

$$\lambda(t|\mathcal{H}_{t-}) = \frac{1}{m}\left(\sum_{k=0}^{N_t-1} \rho^{m-k-1}\lambda_0\left(t - T_{N_k-k}\right) + \sum_{k=N_t}^{m-1} \rho^{m-k-1}\lambda_0(t)\right)$$

$t \in (0, T_1]$		$t \in (T_1, T_2]$		$t \in (T_2, T_3]$		$t \in (T_3, T_4]$
$\frac{1}{m}\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-1}\lambda_0(t-T_1)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-2}\lambda_0(t-T_1)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-3}\lambda_0(t-T_1)$
$\frac{1}{m}\rho\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-1}\lambda_0(t-T_2)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-2}\lambda_0(t-T_2)$
$\frac{1}{m}\rho^2\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\rho\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-1}\lambda_0(t-T_3)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\frac{1}{m}\rho^{m-1}\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-2}\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-3}\lambda_0(t)$	$\longrightarrow$	$\frac{1}{m}\rho^{m-4}\lambda_0(t)$

The exponential smoothing of intensity model

- Then for $1 \le N_t < m$, we have

$$\lambda(t|\mathcal{H}_{t-}) = \frac{1}{m}\left(\sum_{k=0}^{N_t-1} \rho^{m-k-1}\lambda_0\left(t - T_{N_k-k}\right) + \sum_{k=N_t}^{m-1} \rho^{m-k-1}\lambda_0(t)\right)$$

- Then for $N_t \ge m$, we have

$$\lambda(t|\mathcal{H}_{t-}) = \frac{1}{m}\sum_{k=0}^{m-1} \rho^{m-k-1}\lambda_0\left(t - T_{N_k-k}\right)$$

$t \in (T_{N_t-1}, T_{N_t}]$		$t \in (T_{N_t}, T_{N_t+1}]$		$t \in (T_{N_t+1}, T_{N_t+2}]$
$\frac{1}{m}\lambda_0(t - T_{N_t-m})$	$\longrightarrow$	$\frac{1}{m}\rho^{m-1}\lambda_0(t - T_{N_t})$	$\longrightarrow$	$\frac{1}{m}\rho^{m-2}\lambda_0(t - T_{N_t})$
$\frac{1}{m}\rho\lambda_0(t - T_{N_t-m+1})$	$\longrightarrow$	$\frac{1}{m}\lambda_0(t - T_{N_t-m+1})$	$\longrightarrow$	$\frac{1}{m}\rho^{m-1}\lambda_0(t - T_{N_t+1})$
$\frac{1}{m}\rho^2\lambda_0(t - T_{N_t-m+2})$	$\longrightarrow$	$\frac{1}{m}\rho\lambda_0(t - T_{N_t-m+2})$	$\longrightarrow$	$\frac{1}{m}\lambda_0(t - T_{N_t-m+2})$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\frac{1}{m}\rho^{m-1}\lambda_0(t - T_{N_t-1})$	$\longrightarrow$	$\frac{1}{m}\rho^{m-2}\lambda_0(t - T_{N_t-1})$	$\longrightarrow$	$\frac{1}{m}\rho^{m-3}\lambda_0(t - T_{N_t-1})$

The exponential smoothing of intensity model

$$\lambda(t|\mathcal{H}_{t-}) = \begin{cases} \frac{1}{m}\sum_{k=0}^{m-1}\rho^{m-k-1}\lambda_0(t), & \text{if } N_t = 0, \\ \frac{1}{m}\left(\sum_{k=0}^{N_t-1}\rho^{m-k-1}\lambda_0(t-T_{N_t-k}) + \sum_{k=N_t}^{m-1}\rho^{m-k-1}\lambda_0(t)\right), & \text{if } 1 \le N_t < m, \\ \frac{1}{m}\sum_{k=0}^{m-1}\rho^{m-k-1}\lambda_0(t-T_{N_t-k}), & \text{if } N_t \ge m. \end{cases}$$

- In this model, we have parameter ρ and parameters in $\lambda_0(t)$. $\lambda_0(t)$ may be the power law, for example,

$$\lambda_0(t) = \alpha\beta t^{\beta-1}$$

on which the number of parameters in total is 3

The number of parameters

Denote $G_k(t)$ as the probability distribution of time to the k-th failure, T_k, of the component with intensity function $\lambda_0(t)$.

Lemma 1 *For a given m, if $\int_0^\infty \lambda_0(u)dG_k(u) < \infty$, then $E[\lambda(t|\mathscr{H}_{t-})] < \infty$ for $t \to \infty$.*

Lemma 2 *$\lambda(t|\mathscr{H}_{t-})$ may show monotonicity under some conditions, as shown below.*

- *If $\lambda_0(t)$ is an increasing function in t, then $\lambda(t|\mathscr{H}_{t-})$ is an increasing function for $t \in (T^+_{N_t}, T^-_{N_t+1})$;*
- *If $\lambda_0(t)$ is a decreasing function in t, then $\lambda(t|\mathscr{H}_{t-})$ is a decreasing function for $t \in (T^+_{N_t}, T^-_{N_t+1})$.*

The moving average of intensity model

- If $\rho = 1$, then

$$\lambda(t|\mathscr{H}_{t-}) = \begin{cases} \lambda_0(t), & \text{if } N_t = 0, \\ \dfrac{1}{m}\left(\displaystyle\sum_{k=0}^{N_t-1} \lambda_0(t - T_{N_t-k}) + (m - N_t)\lambda_0(t)\right), & \text{if } 1 \le N_t < m, \\ \dfrac{1}{m}\displaystyle\sum_{k=0}^{m-1} \lambda_0(t - T_{N_t-k}), & \text{if } N_t \ge m. \end{cases}$$

Lemma 3

- *If $\lambda_0(t)$ is an increasing function in t, then $\lambda(t|\mathscr{H}_{t-}) \le \lambda_0(t)$;*
- *If $\lambda_0(t)$ is an decreasing function in t, then $\lambda(t|\mathscr{H}_{t-}) \ge \lambda_0(t)$;*
- *If $\lambda_0(t) = \lambda_0$, where λ_0 is a constant, then $\lambda(t|\mathscr{H}_{t-}) = \lambda_0$.*

Comparison

- RP (renewal process)
- NHPP-PL (non-homogeneous Poisson process with the power law)
- GP (geometric process)
- Kijima I (virtual age process I)
- Kijima II (virtual age process II)
- ARI_m (Arithmetic Reduction of Intensity model)
- ARA_m (Arithmetic Reduction of Age model)
- BBIP(Bounded Bathtub Intensity Process)
- Model II (Wu & Scarf, 2017)
- ESI (Exponential Smoothing of Intensity model)
- MAI (Moving Average of Intensity model)

Number of parameters and performance metrics

- The numbers of parameters

Table 2: The number of unknown parameters in each model.

	RP	NHPP-PL	GP	Kijima I	Kijima II	ARI_m	ARA_m	BBIP	Model II	ESI	MAI
q	2	2	3	3	3	3	3	4	4	3	2

- Performance metrics

$$\mathrm{AIC} = -2\log(L) + 2q,$$

$$\mathrm{AIC}_c = -2\log(L) + 2q + \frac{2(q+2)(q+3)}{n-q-2},$$

$$\mathrm{BIC} = -2\log(L) + q\log(n),$$

n: the number of systems; m: the number of components in each system

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Table 3: The means and standard deviations of $(-\log(L))$ from 30 repetitions.

		Estimated values of $(-\log(L))$										
		Nine existing models									New models	
		$q=2$		$q=3$					$q=4$		$q=3$	$q=2$
n	m	RP	NHPP-PL	GP	Kijima I	Kijima II	ARI_m	ARA_m	BBIP	Model II	ESI	MAI
15	5	45.33 (5.07)	42.40 (4.47)	43.08 (4.21)	41.65 (4.25)	41.17 (4.35)	40.82 (4.21)	41.20 (4.29)	42.18 (4.21)	40.60 (4.23)	40.82 (4.25)	41.00 (4.28)
	15	28.73 (4.21)	24.79 (3.38)	26.72 (3.23)	24.18 (3.57)	23.96 (3.68)	24.15 (3.72)	24.25 (3.72)	24.32 (3.23)	23.14 (3.65)	23.84 (3.77)	24.02 (3.67)
	25	23.16 (3.95)	19.35 (3.58)	21.62 (3.04)	18.32 (3.84)	18.33 (3.74)	18.36 (3.83)	18.31 (3.80)	18.96 (3.47)	17.53 (3.96)	18.01 (3.89)	18.24 (3.87)
30	5	91.80 (9.10)	86.67 (8.13)	88.15 (8.10)	86.02 (8.08)	85.46 (7.99)	85.30 (8.33)	85.51 (8.16)	86.34 (8.10)	85.09 (8.10)	85.22 (8.06)	85.33 (8.17)
	15	57.69 (7.70)	50.53 (6.03)	53.60 (5.99)	49.96 (6.19)	49.34 (6.14)	50.02 (6.15)	49.92 (6.12)	49.98 (5.98)	48.42 (6.23)	49.06 (6.13)	49.23 (6.18)
	25	45.03 (7.32)	36.95 (5.57)	41.76 (5.45)	36.46 (5.66)	35.97 (5.81)	36.26 (5.92)	36.26 (5.79)	36.52 (5.44)	35.22 (5.83)	35.67 (5.93)	35.84 (5.95)
45	5	136.12 (13.97)	127.52 (11.82)	130.17 (12.19)	126.61 (11.69)	125.76 (11.63)	125.60 (11.75)	125.71 (11.66)	127.20 (11.84)	125.15 (11.31)	125.41 (11.68)	125.56 (11.62)
	15	84.94 (11.66)	74.17 (7.87)	79.01 (8.77)	73.65 (7.95)	73.12 (8.75)	73.99 (8.24)	73.52 (7.93)	73.49 (7.86)	72.15 (7.73)	72.75 (8.06)	72.96 (8.02)
	25	64.92 (12.03)	52.24 (6.77)	59.85 (9.13)	51.72 (6.94)	51.31 (7.07)	51.69 (7.20)	51.62 (7.11)	51.68 (6.71)	50.57 (7.00)	51.09 (7.02)	51.30 (7.09)

Real world cases

Table 4: The real-world datasets.

No.	Dataset	n	Data source	Model
1	Hydraulic system (LHD 1)	23	Kumar and Klefsjö (1992)	NHPP-PL
2	Hydraulic system (LHD 3)	25	Kumar and Klefsjö (1992)	NHPP-PL
3	Hydraulic system (LHD 9)	27	Kumar and Klefsjö (1992)	NHPP-PL
4	Hydraulic system (LHD 11)	28	Kumar and Klefsjö (1992)	NHPP-PL
5	Hydraulic system (LHD 17)	26	Kumar and Klefsjö (1992)	NHPP-PL
6	Hydraulic system (LHD 20)	23	Kumar and Klefsjö (1992)	NHPP-PL
7	Air conditioner (TBF 7909)	24	Proschan (1963)	HPP
8	Air conditioner (TBF 7912)	30	Proschan (1963)	HPP
9	Air conditioner (TBF 7913)	27	Proschan (1963)	HPP
10	Air conditioner (TBF 7914)	23	Proschan (1963)	HPP
11	Compressor	24	Yanez et al. (2002)	Kijima model I
12	Main propulsion motor	24	Yanez et al. (2002)	Kijima I
13	Powertrain System 510	55	Guida and Pulcini (2009)	BBIP
14	Powertrain System 514	35	Guida and Pulcini (2009)	BBIP
15*	Diesel engine	56	Lee (1980)	NHPP-WLL

* In dataset 15, there is a value 0, which is replaced with 0.5 in this paper.

Model performance on the real-world datasets

Table 5: $-\log(L)$ of each model on the real-world datasets.

	Estimated value of $(-\log(L))$										
	Nine existing models									New models	
	$q=2$		$q=3$					$q=4$		$q=3$	$q=2$
No.	RP	NHPP	GP	Kijima I	Kijima II	ARI_m	ARA_m	BBIP	Model II	ESI	MAI
1	129.99	128.50	129.50	128.46	128.50	128.44	128.46	129.09	128.02	128.32	129.38
2	148.72	146.96	148.72	146.96	145.47	145.31	145.32	146.22	144.94	144.20	144.94
3	166.55	163.64	165.39	163.52	163.65	163.48	163.49	164.36	163.96	163.45	163.97
4	158.05	157.09	157.99	157.09	155.89	155.54	155.86	156.23	155.21	154.91	155.44
5	151.20	149.33	150.96	149.32	149.12	148.38	148.36	149.81	148.98	148.87	149.14
6	137.27	136.86	137.12	136.73	135.65	135.55	135.65	136.61	134.77	135.29	135.80
7	125.37	126.30	124.48	125.37	125.37	125.05	125.37	125.96	125.37	125.37	125.37
8	151.94	150.43	151.14	150.41	150.42	150.37	150.42	150.64	150.20	150.24	151.33
9	143.96	144.22	143.10	143.96	143.10	141.19	142.75	143.81	141.80	142.07	142.15
10	119.60	119.66	119.21	119.52	119.56	118.68	119.57	119.47	117.98	118.58	119.55
11	191.06	189.30	190.95	189.32	188.90	188.70	187.82	189.22	188.78	188.12	188.85
12	183.88	182.44	182.70	181.63	182.45	181.55	181.85	183.29	182.37	182.45	182.91
13	543.26	543.57	543.19	542.28	541.88	543.32	543.58	542.52	542.35	542.35	542.87
14	356.18	357.05	356.06	355.88	355.07	354.55	357.06	353.77	354.74	353.98	355.04
15	369.29	368.31	369.14	368.31	368.00	367.06	367.68	367.70	367.89	367.75	368.07
$-\log(L)$*	205.09	204.24	204.64	203.92	203.53	203.14	203.55	203.91	203.16	203.06	203.65
AIC*	414.17	412.48	415.29	413.83	413.07	412.29	413.10	415.83	414.32	412.13	411.31
AIC_c*	415.20	413.50	417.07	415.62	414.85	414.08	414.88	418.64	417.13	413.91	412.33
BIC*	416.89	415.19	419.36	417.90	417.14	416.36	417.16	421.25	419.74	416.20	414.02

* The value with * on its right upper corner represents the mean of the value.

Results

Table 6: Results of the performance comparison from Table 5.

No.	ESI: $-\log(L)$	ESI: $-\log(L)$ of the "$q=3$" models	MAI: AIC	MAI: AIC_c & BIC
1	Model II	ESI	NHPP-PL	NHPP-PL
2	ESI	ESI	MAI	MAI
3	ESI	ESI	NHPP-PL	NHPP-PL
4	ESI	ESI	MAI	MAI
5	ARA_m	ARA_m	MAI	MAI
6	Model II	ESI	MAI	MAI
7	GP	GP	MAI	MAI
8	Model II	ESI	NHPP-PL	NHPP-PL
9	ARI_m	ARI_m	MAI	MAI
10	Model II	ESI	MAI	MAI
11	ARA_m	ARA_m	MAI	MAI
12	ARI_m	ARI_m	NHPP-PL	NHPP-PL
13	Kijima II	Kijima II	MAI	MAI
14	BBIP	ESI	ESI	MAI
15	ARI_m	ARI_m	MAI	MAI
Frequency	4×Model II	8×ESI	10×MAI	11×MAI

Thank you.

Questions?