A PERTURBATION-INCREMENTAL METHOD FOR
DELAY DIFFERENTIAL EQUATIONS

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A perturbation-incremental (PI) method is presented for the computation, continuation and bi-
furcation analysis of periodic solutions of nonlinear systems of delay differential equations (DDEs).
Periodic solutions can be calculated to any desired degree of accuracy and their stabilities are de-
termined by the Floquet theory. Branch switching at a period-doubling bifurcation is made simple
by the present scheme as a parameter is simply increased from zero to a small positive value so that
a solution on the new branch is obtained. Subsequent continuation of an emanating branch is also
discussed. The advantage of the PI method lies in its simplicity and ease of implementation.

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1. Introduction

Over the past decade, much attention has been paid to the study of delay differential equations (DDEs) in science and engineering such as electronics [Moiola & Chen, 1997; Lu & He, 1996; Chen et al., 1999], optics [Heil et al., 2003; Lepri et al., 1994], neural network [Bélair et al., 1996], biology [Huang, 1993], mechanics [Zhang, 1993; Olgac et al., 1996; Nayfeh et al., 1997], building and structures [Stepan & Haller, 1995]. Exciting applications are also found in the area of controlling chaos by time-delayed feedback control [Pyragas, 1995; Just et al., 1997].

The investigation of periodic solutions and bifurcations of DDEs is rather difficult as such systems are inherently of infinite dimension. Periodic solutions can be obtained through Hopf bifurcation of a steady state solution by means of the center manifold theory [Hale & Verduyn Lunel, 1993; Diekmann et al., 1995]. However, it is in general complicated to reduce a given DDE to a finite dimensional system. Many mathematical models with time delays cannot be treated by existing analytical methods. Moreover, most analytical techniques such as the methods of averaging [Chow & Mallet-Paret, 1997; Claeyssen, 1980] and Liapunov-Schmidt [Fernandes de Oliveira & Hale, 1980; Franke & Stech, 1990; Stech, 1985] are well suited only for bifurcation analysis of specific applications but cannot be applied to a general DDE.

Numerical methods are effective alternative for the investigation of the stability, parameter dependence of periodic solutions and bifurcation of a general DDE. In the shooting approach [Hadeler, 1980; Luzyanina et al., 1997], a DDE is first approximated by a finite dimensional system through the discretisation of an initial function on the delay interval. Then, by using the discrete version of the Poincaré operator, the segment of a periodic solution on the delay interval and its period are calculated. Such an approach usually requires the solution of a large nonlinear system, resulting in expensive computational cost. However, in Luzyanina et al. [1997], the Newton-Picard algorithm was used to enhance the computational efficiency. On the other hand, collocation methods for systems of
ordinary differential equations (ODEs) can also be extended to the computation of periodic solutions of DDEs [Bellen & Zennaro, 1984; Bader, 1985]. A periodic solution is obtained by solving a periodic two-point boundary value problem. In [Engelborghs et al, 2000], it was shown by investigating several collocation methods how superconvergence at the mesh points can be lost or recovered depending on the DDE model and on the choice of collocation discretization. Finally, in the harmonic balance approach, a periodic solution is approximated by a Fourier series [Doedel & Leung, 1982; Stech, 1987; Castelfrancs & Stech, 1987]. Although this approach is efficient in tracing branches of periodic solutions, unphysical oscillations may occur in a periodic solution due to steep gradients.

In real applications, time delay is easy to be controlled and realized. One of the best approaches to control or create complex dynamical motions is by applying a time delay to a dynamical system. Therefore, the analysis of the complex responses of a nonlinear system due to one or more time delays is vital in the control design. In this paper, we present a new numerical method for the continuation of a system of DDEs where the time delay is mainly used as the bifurcation parameter. This method is an extension of the perturbation-incremental (PI) scheme developed in Chan et al. [1996]. This method is similar to the harmonic balance approach in that a periodic solution is approximated by a Fourier series. However, the main difference is that a nonlinear time \( \varphi \) is introduced in the Fourier series instead of the physical time \( t \). An advantage of the present scheme lies in its simplicity and ease of implementation. For instance, at the branch switching of a period-doubling bifurcation, a parameter is simply increased from zero to a small positive value to obtained a solution on the emanating branch. However, in [Leung & Chui, 1995], the tangent of an emanating branch has to be found at the branch switching of a period-doubling bifurcation. Furthermore, we will show by an example that good accuracy can still be achieved using the PI method even to a periodic solution with steep gradients.

The paper is organized as follows. In Section 2, we describe the procedure of the PI scheme for
a system of DDEs. In Section 3, we consider the computation of the stability of a periodic solution. Two examples are given in Sections 4 and 5, and branch switching at a periodic-doubling bifurcation is also discussed. Section 6 contains some comments and conclusions.

2. Perturbation-Incremental Scheme

We consider the following system of DDEs

$$\frac{dx}{dt} = F(x, x_\tau) \quad (1)$$

where $x \in \mathbb{R}^n$, $x_\tau = x(t - \tau)$ with the time delay $\tau > 0$. Let $x(t, \phi)$ be a solution of (1) at time $t$ where $\phi$ is an initial function which is continuous on $[-\tau, 0]$.

The procedure of the PI method is divided into two steps. The first step is to obtain an initial solution for the continuation of the delay $\tau$ in the second step. If (1) possesses a periodic solution at $\tau = 0$, the procedure described in Chan et al. [1996] and Chung et al. [2002] can be used to calculate an initial solution in the form of a truncated Fourier series. If it does not possess a periodic solution at $\tau = 0$, Hopf bifurcation at an equilibrium point $x_0$ is investigated for the existence of a periodic solution. Linearization of (1) produces the system

$$\frac{dx}{dt} = Cx + Dx_\tau, \quad (2)$$

where $C$ and $D$ are the Jacobians given by $C = \frac{\partial F(x, x_\tau)}{\partial x} \bigg|_{x=x_0}$ and $D = \frac{\partial F(x, x_\tau)}{\partial x_\tau} \bigg|_{x=x_0}$. The solution of the characteristic equation of (2) is nontrivial if and only if

$$|\lambda I - C - De^{-\lambda \tau}| = 0. \quad (3)$$

A Hopf bifurcation occurs at $\tau = \tau_0 > 0$ if and only if $\lambda = iw_0$ where $w_0 > 0$ and $\left. \frac{d \text{Re}(\lambda)}{d\tau} \right|_{\tau=\tau_0} \neq 0$ (see [Hale & Verduyn Lunel, 1993; Diekmann et al., 1995]). Then, an approximation of a small-amplitude periodic solution near the bifurcation point can easily be calculated. This will be described in the illustrative examples in Sections 4 and 5. In the second step, small increment is added to $\tau$ and the
Newton-Raphson (NR) procedure is employed to find a neighbouring solution iteratively and for the continuation. Below we describe this parametric incremental procedure.

We introduce a time transformation of the form

\[
\frac{d\varphi}{dt} = \Phi(\varphi), \quad \Phi(\varphi + 2\pi) = \Phi(\varphi),
\]

(4)

where \(\varphi\) is the new time. In the \(\varphi\) domain, (1) is rewritten as

\[
\Phi x' = F(x, x_\tau),
\]

(5)

where prime denotes differentiation with respect to \(\varphi\).

If (1) possesses a periodic solution at \(\tau = 0\) and \(m\) harmonics provide a sufficiently accurate representation for the neighbouring solution corresponding to \(\tau = \Delta\tau\), \((0 < \Delta\tau \ll 1)\), then a periodic solution can be expressed as

\[
x = \sum_{j=0}^{m} (a_j \cos j\varphi + b_j \sin j\varphi),
\]

(6a)

where \(a_j, b_j \in \mathbb{R}^n\) and \(b_0 = 0\). The numerical values of \(a_j\) and \(b_j\) in (6a) can be obtained from the procedure described in Chan et al. [1996] or Chung et al. [2002], and the continuation in \(\tau\) is started from the large-amplitude periodic solution at \(\tau = 0\). On the other hand, if there is no periodic solution at \(\tau = 0\), the continuation may be started from a Hopf bifurcation at an equilibrium point with \(\tau = \tau_0\). Let the origin be an equilibrium point. The neighbouring solution corresponding to \(\tau = \tau_0 + \Delta\tau\) or \(\tau = \tau_0 - \Delta\tau\) can also be expressed in the form of (6a). The numerical values of \(a_j\) and \(b_j\) can be determined from the first step. Let \(a_{ij}, b_{ij}, x_i \in \mathbb{R} (1 \leq i \leq n, 0 \leq j \leq m)\) be the \(i\)-th element in \(a_j, b_j\) and \(x\), respectively. Initially, \(x_1\) contains only a few harmonics. Therefore, \(x_1\) can be expressed as

\[
x_1 = \sum_{j=0}^{m_1} (a_{1j} \cos j\varphi + b_{1j} \sin j\varphi),
\]

(6b)
where \( m_1 \leq m \). In fact, \( m_1 = 1 \) for a small-amplitude periodic solution near a Hopf bifurcation and the large-amplitude periodic solutions considered in Chan et al. [1996] and Chung et al. [2002]. In the algorithm described below, \( x_1 \) takes the form of (6b).

If \( \varphi_1 \) is the new time corresponding to \( t - \tau \), it follows from (4) that
\[
dt = \frac{d\varphi}{\Phi(\varphi)} = \frac{d\varphi_1}{\Phi(\varphi_1)}
\]
\[
\Rightarrow \Phi(\varphi) \frac{d\varphi_1}{d\varphi} = \Phi(\varphi_1).
\]

(7)

We note that \( \varphi_1 - \varphi \) is a periodic function in \( \varphi \) with period \( 2\pi \). To consider the continuation with the delay \( \tau \) as the bifurcation parameter, a small increment of \( \tau \) to \( \tau + \Delta \tau \) corresponds to small changes of the following quantities
\[
x \rightarrow x + \Delta x, \quad \Phi \rightarrow \Phi + \Delta \Phi \quad \text{and} \quad \varphi_1 \rightarrow \varphi_1 + \Delta \varphi_1.
\]

To obtain a neighbouring solution, (5) and (7) are expanded in Taylor’s series about an initial solution and linearized incremental equations are derived by ignoring all the non-linear terms of small increments as below
\[
x' \Delta \Phi(\varphi) + \Phi(\varphi) \Delta x' - \left. \frac{\partial F(x, x_r)}{\partial x} \right|_0 \Delta x - \left. \frac{\partial F(x, x_r)}{\partial x_r} \right|_0 \Delta x_r
\]
\[
= F(x, x_r) - \Phi(\varphi)x',
\]
\[
\varphi_1' \Delta \Phi(\varphi) + \Phi(\varphi) \Delta \varphi_1' - \Delta \Phi(\varphi_1) - \Phi'(\varphi_1) \Delta \varphi_1 = \Phi(\varphi_1) - \Phi(\varphi)\varphi_1',
\]

(8b)

where the subscript 0 represents the evaluation of the relevant quantities corresponding to the initial solution. From (6), the terms \( \Delta x \) and \( \Delta x' \) are expressed as, respectively,
\[
\Delta x = \sum_{j=0}^{m} (\Delta a_j \cos j\varphi + \Delta b_j \sin j\varphi),
\]
\[
\Delta x' = \sum_{j=1}^{m} j(\Delta b_j \cos j\varphi - \Delta a_j \sin j\varphi).
\]

(9b)
Since $\Phi$ and $\varphi_1 - \varphi$ are both periodic functions in $\varphi$ with period $2\pi$, we write

$$\Phi(\varphi) = \sum_{j=0}^{m} (p_j \cos j\varphi + q_j \sin j\varphi), \quad (9c)$$

$$\Delta \Phi(\varphi) = \sum_{j=0}^{m} (\Delta p_j \cos j\varphi + \Delta q_j \sin j\varphi), \quad (9d)$$

$$\Delta \Phi'(\varphi) = \sum_{j=1}^{m} j (\Delta q_j \cos j\varphi - \Delta p_j \sin j\varphi), \quad (9e)$$

and

$$\varphi_1 = \varphi + \sum_{j=0}^{m} (r_j \cos j\varphi + s_j \sin j\varphi), \quad (9f)$$

$$\Delta \varphi_1 = \sum_{j=0}^{m} (\Delta r_j \cos j\varphi + \Delta s_j \sin j\varphi), \quad (9g)$$

$$\Delta \varphi'_1 = \sum_{j=1}^{m} j (\Delta s_j \cos j\varphi - \Delta r_j \sin j\varphi). \quad (9h)$$

For the delay term $x_\tau$, we have

$$x_\tau = \sum_{j=0}^{m} (a_j \cos j\varphi_1 + b_j \sin j\varphi_1), \quad (9i)$$

$$\Delta x_\tau = \sum_{j=0}^{m} (\Delta a_j \cos j\varphi_1 + \Delta b_j \sin j\varphi_1) + \frac{\partial x_\tau}{\partial \varphi_1} \Delta \varphi_1. \quad (9j)$$

The integration constant of (7) provides information about the delay $\tau$. Since $\varphi_1$ is the new time corresponding to $t - \tau$, it follows from (7) that

$$\int_{t-\tau}^{t} dt_1 = \int_{\varphi_1}^{\varphi} \frac{d\theta}{\Phi(\theta)} \implies \tau = \int_{\varphi_1}^{\varphi} \frac{d\theta}{\Phi(\theta)}. \quad (10)$$

For a small increment of $\tau$ to $\tau + \Delta \tau$, the linearized incremental equation of (10) is given by

$$\int_{\varphi_1}^{\varphi} \frac{\Delta \Phi(\theta)}{\Phi^2(\theta)} d\theta + \frac{\Delta \varphi_1}{\Phi(\varphi_1)} = \int_{\varphi_1}^{\varphi} \frac{d\theta}{\Phi(\theta)} - \tau - \Delta \tau, \quad (11)$$

which implies, for $\varphi = 0$,

$$\int_{\alpha}^{0} \frac{\Delta \Phi(\theta)}{\Phi^2(\theta)} d\theta + \frac{\Delta \varphi_1(0)}{\Phi(\alpha)} = \int_{\alpha}^{0} \frac{d\theta}{\Phi(\theta)} - \tau - \Delta \tau, \quad (12)$$
where $\alpha = \varphi_1(0)$. The harmonic balance method is applied to (8a), (8b) and (12). Rewriting the
linearized equation (8a) in terms of the increments $\Delta a_j, \Delta b_j, \Delta p_j, \Delta q_j, \Delta r_j$ and $\Delta s_j$, we have

$$
\sum_{j=0}^{m} [\Psi_{1,j} \Delta a_j + \Psi_{2,j} \Delta b_j + \Psi_{3,j} \Delta p_j + \Psi_{4,j} \Delta q_j + \\
\Psi_{5,j} \Delta r_j + \Psi_{6,j} \Delta s_j] = \Lambda_1, \quad (13a)
$$

where

$$
\Psi_{1,j} = -j \Phi(\varphi) \sin j\varphi - \frac{\partial F}{\partial x} \bigg|_0 \cos j\varphi - \frac{\partial F}{\partial x_\tau} \bigg|_0 \cos j\varphi_1, \\
\Psi_{2,j} = j \Phi(\varphi) \cos j\varphi - \frac{\partial F}{\partial x} \bigg|_0 \sin j\varphi - \frac{\partial F}{\partial x_\tau} \bigg|_0 \sin j\varphi_1, \\
\Psi_{3,j} = x' \cos j\varphi, \\
\Psi_{4,j} = x' \sin j\varphi, \\
\Psi_{5,j} = -\frac{\partial F}{\partial x_\tau} \bigg|_{0} \frac{\partial x_\tau}{\partial \varphi_1} \cos j\varphi, \\
\Psi_{6,j} = -\frac{\partial F}{\partial x_\tau} \bigg|_{0} \frac{\partial x_\tau}{\partial \varphi_1} \sin j\varphi, \\
\Lambda_1 = F(x, x_\tau) - \Phi(\varphi)x'.
$$

Similarly, from (8b) and (12), we obtain, respectively,

$$
\sum_{j=0}^{m} [\Psi_{7,j} \Delta p_j + \Psi_{8,j} \Delta q_j + \Psi_{9,j} \Delta r_j + \Psi_{10,j} \Delta s_j] = \Lambda_2, \quad (13b)
$$

and

$$
\sum_{j=0}^{m} [\Psi_{11,j} \Delta p_j + \Psi_{12,j} \Delta q_j + \Psi_{13,j} \Delta r_j] = \Lambda_3, \quad (13c)
$$
where

\begin{align*}
\Psi_{7,j} &= \phi'_1 \cos j \phi - \cos j \phi_1, \\
\Psi_{8,j} &= \phi'_1 \sin j \phi - \sin j \phi_1, \\
\Psi_{9,j} &= -j \Phi(\phi) \sin j \phi - \Phi'(\phi_1) \cos j \phi, \\
\Psi_{10,j} &= j \Phi(\phi) \cos j \phi - \Phi'(\phi_1) \sin j \phi, \\
\Psi_{11,j} &= \int_{\alpha}^{0} \frac{\cos j \theta}{\Phi^2(\theta)} d\theta, \\
\Psi_{12,j} &= \int_{\alpha}^{0} \frac{\sin j \theta}{\Phi^2(\theta)} d\theta, \\
\Psi_{13,j} &= \frac{1}{\Phi(\alpha)}, \\
\Lambda_2 &= \Phi(\phi_1) - \Phi(\phi) \phi'_1, \\
\Lambda_3 &= \int_{\alpha}^{0} \frac{d\theta}{\Phi(\theta)} - \tau - \Delta \tau.
\end{align*}

Since $\Psi_{i,j}$ ($1 \leq i \leq 13, 1 \leq j \leq m$) and $\Lambda_k$ ($1 \leq k \leq 3$) are periodic functions in $\phi$, they can be expressed in Fourier series which coefficients can easily be obtained by the method of Fast Fourier Transform (FFT). By comparing the coefficients of $(2m + 1)n + 2m_1 - 1$ harmonic terms of (13a), $2m + 1$ of (13b) and (13c), a system of linear equations is thus obtained with unknowns $\Delta a_{ij}, \Delta b_{ij}, \Delta p_j, \Delta q_j, \Delta r_j$ and $\Delta s_j$ in the form

\begin{equation}
\sum_{i=1}^{n} \sum_{j=0}^{m} (A_{k,ij} \Delta a_{ij} + B_{k,ij} \Delta b_{ij}) + \\
\sum_{j=0}^{m} (P_{k,j} \Delta p_j + Q_{k,j} \Delta q_j + R_{k,j} \Delta r_j + S_{k,j} \Delta s_j) = T_k, \tag{14}
\end{equation}

where $k = 1, 2, \ldots, (2m + 1)(n + 1) + 2m_1$ and $T_k$ are residue terms. The values of $a_j, b_j, p_j, q_j, r_j$ and $s_j$ are updated by adding the original values and the corresponding incremental values. The iteration process continues until $T_k \to 0$ for all $k$ (in practice, $|T_k|$ is less than a desired degree of accuracy). The entire incremental process proceeds by adding the $\Delta \tau$ increment to the converged
value of $\tau$, using the previous solution as the initial approximation until a new converged solution is obtained. In case when a saddle node bifurcation occurs in the continuation, $a_{11}$ will be used as the control parameter.

3. Stability of periodic solution

The stability of a periodic solution can be determined by the Floquet method [Hale & Verduyn Lunel, 1993; Diekmann et al., 1995; Luzyanina et al., 1997]. Let $\zeta \in \mathbb{R}^n$ be a small perturbation from a periodic solution of (1). Then,

$$\frac{d\zeta}{dt} = \frac{\partial F}{\partial x} \zeta + \frac{\partial F}{\partial x_\tau} \zeta_\tau,$$

$$\Rightarrow \frac{d\zeta}{d\varphi} = \frac{1}{\Phi} [A(\varphi, \varphi_1) \zeta + B(\varphi, \varphi_1) \zeta_\tau], \quad (15)$$

where $A(\varphi, \varphi_1) = \frac{\partial F}{\partial x}$ and $B(\varphi, \varphi_1) = \frac{\partial F}{\partial x_\tau}$. The entities of $A$ and $B$ are all periodic functions of $\varphi$ with period $2\pi$, which can be determined by using the incremental procedure. The time delay interval $I_1 = [-\tau, 0]$ corresponds to $I_2 = [\alpha, 0]$ in the $\varphi$ domain. Discrete points in $I_2$ are selected for the computation of Floquet multipliers.

From the incremental procedure, the Fourier coefficients of $\varphi_1$ in (9f) are obtained. Assume that $\varphi = \beta$ when $\varphi_1 = 0$ and let $I_3 = [0, \beta]$. For each $\varphi \in I_3$, there corresponds a unique $\varphi_1 \in I_2$. We choose a mesh size $h = \frac{\beta}{N-1}$ and discrete points $\varphi^{(i)} = ih$ ($0 \leq i \leq N - 1$) in $I_3$, which correspond to $\varphi^{(i)}_1 = \varphi_1(\varphi^{(i)})$ in $I_2$. Let $\zeta^{(i)}(\varphi_1)$ be the $(i + 1)$-th unit vector in $\mathbb{R}^N$. By applying numerical integration to (15), we obtain the monodromy matrix $M$ as

$$M = [\zeta(\varphi_1^{(0)}) + 2\pi, \zeta(\varphi_1^{(1)}) + 2\pi, \ldots, \zeta(\varphi_1^{(N-1)}) + 2\pi]].$$

The eigenvalues of $M$ are used to determine the stability of the periodic solution. One of the eigenvalues or Floquet multipliers of $M$ must be unity which provides a check for the accuracy of the calculation. If all the other eigenvalues are inside the unit circle, the periodic solution under
consideration is stable; otherwise, it is unstable. A period-doubling bifurcation occurs if one of the
eigenvalues crosses the unit circle at $-1$.

4. A model of Recurrent Neural Feedback

Recurrent neural feedback occurs when activity in a population of neurons excites a second population
via axon collaterals and the second population in turn excites or inhibits the first one. Time delays
of recurrent feedback come from the conduction times and synaptic delays in the feedback circuit.
The following two dimensional model was proposed by Plant [1981]

\[
\frac{dv}{dt} = v - w - \frac{v^3}{3} + \mu(v_r - v_0), \\
\frac{dw}{dt} = \rho(v + a - bw),
\]

(16)

where $a$, $b$, $\rho$ are real parameters, $\mu$ measures the strength of the feedback, $v$ and $w$ represent
membrane potential and a recovery valuable, respectively. $v_0$ is the unique root of $\frac{v^3}{3} - v_0 + \frac{(v_0 + a)}{b} = 0$
which together with $w_0 = \frac{(v_0 + a)}{b}$ corresponds to an equilibrium of the model. Periodic solutions and
bifurcations of (16) with respect to the parameter $\mu$ were studied in Castelfranco & Stech [1987] and
Engelborghs et al. [2002].

In this section, we investigate the periodic solutions and bifurcations of (16) with respect to the
time delay $\tau$. It can be seen that the dynamics of the system becomes very complicated as $\tau$ increases
and multiple periodic solutions exist for large $\tau$.

We first consider Hopf bifurcation of (16) in order to obtain an initial guess. Shifting the origin
to the equilibrium point $(v_0, w_0)$, we obtain from (16)

\[
\frac{dv}{dt} = (1 - v_0^2)v - w - v_0v^2 - \frac{1}{3}v^3 + \mu v_r, \\
\frac{dw}{dt} = \rho(v - bw).
\]

(17)
Linearization of (17) produces the system
\[
\frac{dx}{dt} = Cx + Dx_\tau, \tag{18}
\]
where \(x = \begin{pmatrix} v \\ w \end{pmatrix}, C = \begin{pmatrix} 1 - v_0^2 & -1 \\ \rho & -\rho b \end{pmatrix}\) and \(D = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}\).

In Castelfranco & Stech [1987] and Engelborghs et al. [2002], with \(\tau\) fixed, the continuation in \(\mu\) starting from \(\mu = -0.5\) was investigated for the parameters \(a = 0.7, b = 0.8\) and \(\rho = 0.08\), giving \(v_0 = -1.1994\). With the above parameter values, we investigate the continuation in \(\tau\) at \(\mu = -0.5\).

From (3), two Hopf bifurcations \((w_1, \tau_1) = (0.3662, 7.7231)\) and \((w_2, \tau_2) = (0.2819, 11.0598)\) are detected for which \(0 \leq w_i \tau_i < 2\pi\). We first consider the initial guess for the Hopf bifurcation at \(\tau = \tau_1\).

Since \(\frac{dx}{dt} = \Phi(\varphi) = w_1\), we have from (9c)
\[p_0 = w_1 = 0.3662, p_j = q_j = 0 \quad \text{for} \quad j \geq 1.\]

From (10), we have \(\varphi_1 = \varphi - w_1 \tau_1 = \varphi - 2.8281\). It follows from (9f) that
\[r_0 = -2.8281, r_j = s_j = 0 \quad \text{for} \quad j \geq 1.\]

Since a periodic solution near the Hopf bifurcation at \(\tau = \tau_1\) has a very small amplitude, arbitrary small values may be initially assigned to \(a_j\) and \(b_j\) in (6a) for \(m_1 = 1\) and small \(m\). By applying the incremental procedure to \(\tau'_1 = \tau_1 + \varepsilon\) where \(|\varepsilon| \ll 1\), a converged periodic solution at \(\tau = \tau'\), if it exists, can easily be obtained after a few iterations.

Fig. 1 shows the continuation curves of (17) from the Hopf bifurcations \(\tau = \tau_1 = 7.7231\) and \(\tau = \tau_2 = 11.0598\). Observe that an unstable subcritical Hopf bifurcation occurs at \(\tau = \tau_1\) (label 1).

As \(\tau\) varies, the periodic solution undergoes saddle-node bifurcations, period-doubling bifurcations and ends up at \(\tau = 29.5364\) (label 6) which is a period halving. Next, the Hopf bifurcation occurred at \(\tau = \tau_2\) (label 7) is supercritical. The periodic solution remains unstable as \(\tau\) varies and ends up at \(\tau = 19.9329\) (label 8) which is also a period halving.
Fig. 1. Continuation curves of (17) starting from the Hopf bifurcations at $\tau = \tau_1$ and $\tau = \tau_2$. ■, Hopf bifurcation; ▲, Period-doubling bifurcation; △, Period halving; —, stable; - - -, unstable.

Let the family of orbits on the continuation curve starting from $\tau = \tau_1$ be denoted by $F_1$. Then, an infinite number of families $F_i$ ($i = 2, 3, \cdots$) of orbits can be obtained from $F_1$. Assume that $x^*(t)$ is a periodic solution of (1) with period $T^*$. Then, $x^*_{\tau} = x^*_{\tau + T^*}$. It follows that if $(v_{\text{max}}, \tau)$ in Fig. 1 represents a periodic solution of (17) with period $T^*$, so do the points $(v_{\text{max}}, \tau + nT^*)$ for $n = 2, 3, \cdots$. The period of a periodic solution in $F_1$ can be obtained from (7a) as $T = \int_0^{2\pi} \frac{d\varphi}{\varphi(\varphi)}$. Fig. 2 shows the families $F_i$ ($i = 1, 2, \cdots, 7$) of orbits having Hopf bifurcations at $\tau_i = \tau_1 + \frac{2\pi(i-1)}{w_1}$. Corresponding periodic solutions represented by the points $(v_{\text{max}}, \tau + nT^*)$ with different $n$ have the same phase portrait in phase space but their stabilities are all different. In the stability calculation of a periodic solution as outlined in Section 3, the mesh size is fixed at $h = 0.01$ for convenience. The number of mesh points can be determined once $\beta$ in $I_3$ is found. Figs. 3 and 4 show the behaviour of the Floquet multipliers along $F_1$ and $F_2$ respectively.

The period halvings at labels 6 and 8 in Fig. 1 are actually period-doubling points in $F_2$. Therefore, the continuation curve between labels 7 and 8 can be regarded as an emanating branch of $F_2$ from a period-doubling bifurcation. Once a period-doubling bifurcation occurs in $F_i$, corresponding
period-doubling bifurcations can be found in $F_{i+2j}$ for $j = 1, 2, \cdots$ (see labels 9 and 10 of Fig. 2). Therefore, emanating branches accumulate in $F_i$ as $i$ increases.

![Figure 2](image_url)  

**Fig. 2.** Continuation curves of the families $F_1$-$F_7$. ▼, Hopf bifurcation; ▲, Period-doubling bifurcation; •, Torus bifurcation; ---, stable; - - -, unstable.

Branch switching at a period-doubling bifurcation can easily be achieved by using the PI scheme as described in Chung et al. [2003]. For instance, at label 2 of Fig. 1 where a period-doubling bifurcation occurs, the periodic solution is expressed in the form of (6a), (9c) and (9f) where the numerical values of $a_j$, $b_j$, $p_j$, $q_j$, $r_j$, $s_j$ and the dominant Floquet multipliers are given in Table 1. To obtain a period-2 periodic solution on the emanating branch, we rescale $\varphi$ to $2\varphi$ and replace the original solution with $m$ harmonics as

$$x = \sum_{j=0}^{2m} (a_j^* \cos j\varphi + b_j^* \sin j\varphi),$$

$$\Phi(\varphi) = \sum_{j=0}^{2m} (p_j^* \cos j\varphi + q_j^* \sin j\varphi),$$

$$\varphi_1 = \varphi + \sum_{j=0}^{2m} (r_j^* \cos j\varphi + s_j^* \sin j\varphi),$$
where \( a^*_{2j} = a_j, b^*_{2j} = b_j, p^*_{2j} = \frac{p_j}{2}, q^*_{2j} = \frac{q_j}{2}, r^*_{2j} = \frac{r_j}{2}, s^*_{2j} = \frac{s_j}{2} \) for \( j = 0, 1, 2, \cdots, m \) and 
\( a^*_j = b^*_j = p^*_j = q^*_j = r^*_j = s^*_j = 0 \) for odd \( j \). For a nearby periodic solution on the emanating branch from a period-doubling bifurcation, \( a^*_j \) and \( b^*_j \) with odd \( j \) will not be all zero vectors. Similar to Chung et al. [2003], \( b_{11} \) is chosen as the continuation parameter which is simply turned on from zero to a small positive value in order to obtain a solution on the emanating branch. In this way, the tedious computations from traditional branch-switching methods involving the tangent of the new branch and second derivatives are all avoided [Seydel, 1994; Nayfeh & Balachandran, 1995].
Emanating branches from period-doubling bifurcation between labels 2 and 3, and between labels 4 and 5 are shown in Figs. 5 and 6, respectively. Four period-doubling bifurcations are found on the emanating branch of Fig. 5. Figures 7 and 8 show the corresponding emanating branch on $F_3$ and the behaviour of the dominant Floquet multipliers, respectively. Three torus bifurcations are found instead in that branch. From Fig. 2, torus bifurcation occurs frequently in $F_i$ as $i$ increases and, thus, the dynamics become quite complicated for large $\tau$. Further, we observe in Fig. 2 that three stable periodic solutions coexist in the intervals $\tau \in [29.4954, 32.2003]$ and $\tau \in [74.4311, 84.0981]$. 

Fig. 4. (a) Moduli of the computed multipliers along the branch of periodic solutions $F_2$ ■, Hopf bifurcation; ▲, Period-doubling bifurcation; ●, Torus bifurcation. (b) Two enlargements of Fig. 4(a).
\( m = 10, \ m_1 = 1, \ a_{10} = 0.24211, \ a_{11} = 0.89273, \ \tau = 5.37521 \)

Floquet multipliers: \(-3.20781; -1.00058; 0.99970; 0.01383 \pm 0.01094i\)

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Table 1. Fourier coefficients of \( x_1, x_2, \Phi(\varphi) \) and \( \varphi_1 \) for the periodic solution at label 2 of Fig. 1.

\[ Max(v) \]

Fig. 5. Emanating branch from a period-doubling bifurcation at \( \tau \simeq 5.37521 \) (label 2) of Fig. 1. \( \blacktriangle \), Period-doubling bifurcation; —, stable; - - -, unstable.
Fig. 6. Emanating branch from a period-doubling bifurcation at $\tau \simeq 30.9042$ (label 4) of Fig. 1. ▲, Period-doubling bifurcation; ---, stable; -- --, unstable.

Fig. 7. Emanating branch on $F_3$ corresponding to the one shown in Fig. 5. ▲, Period-doubling bifurcation; ●, Torus bifurcation; ---, stable; -- --, unstable.
Fig. 8. Moduli of the computed multipliers along the emanating branch shown in Fig. 7. ● Torus bifurcation.

From the PI method, an explicit form of a periodic solution can be obtained for arbitrary $\tau$. For instance, for $\tau = 6.3811$ (label 11) on the emanating branch of Fig. 5, the explicit form of the stable periodic solution is given in the form of (6a), (9c) and (9f) where the numerical values of $a_j$, $b_j$, $p_j$, $q_j$, $r_j$, $s_j$ and the dominant Floquet multipliers are given in Table 2. A phase portrait of the stable period solution is shown in Fig. 9 and is compared to the result of the numerical integration obtained by using the fourth order Runge-Kutta method. For $\tau = 6$ (label 12) on the emanating branch of Fig. 5, the periodic solution is unstable and its information is given in Table 3. The phase portrait is compared with the result from a shooting method as shown in Fig. 10. In both cases, the results are in good agreement.
\[ m = 20, \ m_1 = 2, \ \tau = 6.3811 \]

\[ a_{10} = 0.54164, \ a_{11} = -0.95841, \ a_{12} = 1.28922, \ b_{11} = 0.27778 \]

Floquet multipliers: 0.99994; \(-0.146458 \pm 0.14960i; \ -0.00038 \pm 0.00063i\)

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Table 2. Fourier coefficients of \(x_1, x_2, \Phi(\varphi)\) and \(\varphi_1\) for the periodic solution at label 11 of Fig. 5.

Fig. 9. Periodic solution at \(\tau = 6.3811\) (label 11) on the emanating branch of Fig. 5. —, Runge-Kutta method; \(\times\), Perturbation-Incremental method.
\[ m = 10, \quad m_1 = 2, \quad \tau = 6 \]

\[ a_{10} = 0.33570, \quad a_{11} = -0.55735, \quad a_{12} = 1.0120, \quad b_{11} = 0.19024 \]

Floquet multipliers: 

\(-491.043; 1.00000; 0.02268; 0.00039 \pm 0.00022i\)

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Table 3. Fourier coefficients of \(x_1, x_2, \Phi(\varphi)\) and \(\varphi_1\) for the periodic solution at label 12 of Fig. 5.

Fig. 10. Periodic solution at \(\tau = 6\) (label 12) on the emanating branch of Fig. 5. - - -, Shooting method; \(	imes\), Perturbation-Incremental method.

**Unphysical oscillations may occur in a computed periodic solution with steep gradients obtained by using a Fourier-based method. We test the performance of the PI method on the computation of**
such a periodic solution at $\tau = 25$ and $\mu = -2$. In [Engelborghs et al, 2002], the periodic solution was constructed by using the software package DDE-BIFTOOL with different uniform and adapted meshes. Unphysical oscillations near steep gradients are computed on the uniform mesh when the subinterval size is not fine enough. In comparison, convergence of the period and dominant Floquet multipliers for different harmonic size $M$ is illustrated in Table 4. It can be seen from Fig. 11 that, unphysical oscillation is not apparent in the computed periodic solution even for small $M(= 20)$.

<table>
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<th>$T$</th>
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Table 4. Convergence of the period and dominant Floquet multipliers for different harmonic size $M$.

Fig. 11. Periodic solution at $\tau = 25$ and $\mu = -2$.

5. The generalized van der Pol oscillator
Next, we consider the delayed generalized van der Pol oscillator of the form

\[
\dot{v} = w,
\]

\[
\dot{w} = -v - v^2 + \lambda[(\alpha + v - v^2)w + \beta(v - v_\tau) + \gamma(v - v_\tau)^3],
\]  

(19)

where \(\lambda, \alpha, \beta, \gamma\) are real parameters. Without delay (i.e. \(\tau = 0\)), it has been shown in Chan et al. [1996] that system (19) possesses a periodic solution for \(\alpha = 0.1\). We investigate the dynamics of (19) when \(\lambda = 3, \alpha = 0.1, \beta = 0.3\) and \(\gamma = -1\). Continuation of (19) can be started from either the periodic solution without delay obtained by using the PI scheme described in Chan et al. [1996] or a Hopf bifurcation obtained from the linearization of (19). For the former case, the periodic solution is given by

\[
v = 0.5091 \cos \varphi - 0.0589
\]

\[
w = \dot{v} = -0.5091 \Phi(\varphi) \sin \varphi
\]

where the Fourier coefficients of \(\Phi(\varphi)\) are given in Table 5. Since \(\tau = 0\), we have

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Table 5. Fourier coefficient of \(\Phi(\varphi)\) at \(\tau = 0\) for Equations (19).

\(\varphi_1 = \varphi\), i.e. \(r_j = s_j = 0\). Further, the period of this solution is \(T^* = 8.0959\). For the latter case, it follows from (2), (3) and (19) that two Hopf bifurcations \((\omega_1, \tau_1) = (0.6962, 5.6189)\) and \((\omega_2, \tau_2) = (0.9085, 5.6616)\) where \(0 \leq \omega_i \tau_i < 2\pi\) are obtained. Starting from the periodic solution
without delay, the stable periodic solution undergoes two saddle-node bifurcations and vanishes at the Hopf bifurcation with $\tau = \tau_1$, as shown in Fig. 12.

![Max(v)](image)

Fig. 12. Continuation curve of (19) starting from the periodic solution without delay, i.e. $\tau = 0$. ■, Hopf bifurcation; —, stable; - - -, unstable.

We denote this family of orbits by $F_1$. As is shown in the last section, $F_1$ corresponds to an infinite number of families $F_i$ where $i = 2, 3, \ldots$. The corresponding periodic solutions in $F_2$ are shown in the continuation curve of Fig. 13 between $\tau = T^*$ (label 1) and the Hopf bifurcation at $\tau = \tau_1 + T_1 = 14.6444$ where $T_1 = 9.0255$ is the period of the Hopf bifurcation at $\tau = \tau_1$. As $\tau$ decreases from $T^*$, the curve ends at $\tau = \tau_2$. The period of the Hopf bifurcation at $\tau = \tau_2$ is $T_2 = 6.9160$. Two period-doubling bifurcations (labels 2 and 3) and two torus bifurcations are detected on this continuation curve. Fig. 14 shows the behaviour of the dominant Floquet multipliers along $F_2$. On the emanating branch from label 2, four period-doubling bifurcations (labels 4-7) are obtained and the periodic solutions are mostly unstable except three short stable segments: between labels 2 and 4, labels 3 and 7, labels 6 and 8 (which is a saddle-node bifurcation). We also note that there is no periodic solution in the region $\tau \in (\tau_1, \tau_2)$ which is a so-called amplitude death region
where any vibration dies down eventually [Reddy et al., 2000; Xu & Chung, 2003]. This suggests that the vibration of (19) without delay, i.e. $\tau = 0$, can be brought to a standstill if an appropriate position time delay feedback is added to the system. Physically, amplitude death region is very useful in controlling the stability of a system.

The continuation curve of Fig. 13 indicates that there is always a continuation curve connecting the Hopf bifurcations $\tau = \tau_2 + (n - 2)T_2$ and $\tau = \tau_1 + (n - 1)T_1$ for $n = 2, 3, \cdots$. The corresponding continuation curves for $n = 3, 4, 5$ are shown in Fig. 15. We observe that the occurrence of torus bifurcation on the continuation curves becomes more frequent as $n$ increases. Fig. 16 shows the emanating branch from the period-doubling bifurcation at $\tau = 19.9346$ on $F_4$, (label 9 of Fig. 15), which corresponds to that from label 2 of Fig. 13. Interestingly, three torus bifurcations are also found on this branch in addition to period-doubling bifurcation.

From the PI method, explicit form of a periodic solution can be obtained for arbitrary $\tau$. For instance, for $\tau = 33.2439$ (label 10) on $F_5$ of Fig. 15, the explicit form of the stable periodic solution

\[ Max(v) \]

\[ \tau \]

Fig. 13. Continuation curve of (19) starting from $\tau = \tau_2$. ■, Hopf bifurcation. ▲, Period-doubling bifurcation; ●, Torus bifurcation; —, stable; -- --, unstable.
Fig. 14. Moduli of the computed multipliers along the branch of periodic solutions $F_2$ ■, Hopf bifurcation; ▲, Period-doubling bifurcation. ▼, Period-doubling bifurcation; ●, Torus bifurcation.

Max(\(v\))

Fig. 15. Continuation curves of (19) starting from $\tau = \tau_2 + (n - 1)T_2$ where $n = 2, 3, 4$. ■, Hopf bifurcation; ▲, Period-doubling bifurcation; ●, Torus bifurcation; —, stable; - - -, unstable.
is given in the form of (6a), (9c) and (9f) where the numerical values of $a_j$, $b_j$, $p_j$, $q_j$, $r_j$ and $s_j$ are given in Table 6. Phase portrait of the period solution is shown in Fig. 17 and is compared to the result of the numerical integration obtained by using the fourth order Runge-Kutta method. It can be seen that they are in good agreement.
$m = 10$, $m_1 = 1$, $a_{10} = -0.08154$, $a_{11} = 0.43399$, $\tau = 33.2439$

Floquet multipliers: 1.00004; $0.61409 \pm 0.74655i$; $0.89611 \pm 0.28558i$

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Table 6. Fourier coefficients of $x_1$, $x_2$, $\Phi(\varphi)$ and $\varphi_1$ for the periodic solution at label 10 of Fig. 15.

Fig. 17. Periodic solution at $\tau = 33.2439$ (label 10) on $F_5$ of Fig. 15. —, Runge-Kutta method; $\times$, Perturbation-Incremental method.

6. Conclusion

The Perturbation-Incremental (PI) method is extended for the computation, continuation and bifurcation analysis of periodic solutions of nonlinear systems of delay differential equations (DDEs)
where the delay is used as the bifurcation parameter. Thus, the complex responses of a nonlinear system due to time delay can be observed and analysed. An explicit form of a periodic solution with arbitrary time delay value can be obtained, which enables the phase portrait to be constructed and its stability to be determined by using the Floquet method. The stable (unstable) periodic solutions are in good agreement with those obtained from the Runge-Kutta method (a shooting method). The calculation for switching branches at a period-doubling bifurcation is made simple since the heavy computation of the tangent of a new branch and second derivatives are all avoided.

A model of recurrent neural feedback and the generalized van der Pol oscillator are used as illustrative examples. Both results show that torus bifurcation occurs frequently for large \( \tau \) and thus induces complicated dynamical behaviours. The present algorithm can also be extended further to DDEs with multiple time delays, state-dependent delays and of neutral types. The advantage of the PI method lies in its simplicity and ease of implementation.

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References

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