A Unified Theory for Surface and Volumetric Growths of Biological Tissues

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Soft and hard biological tissues may grow in different manners. Mathematically, volumetric growth refers to the addition of material points throughout the volume, while surface growth refers to the addition of material points on a surface. Currently, these two types of growths are modeled with different mathematical formulations and analyses. In particular, a clean and consistent theory for surface growth is yet to be developed, hampered by the difficulties associated with the change of topology of a set of material points.

We present an approach which could lead to a growth theory that describes both surface and volumetric growths in a unified framework. We start by formulating volumetric growth by using both the material description and the spatial description. In the former, the independent variables are time and the position of material points in a fixed reference configuration, and in the latter, are time and the position in space. While the material description is commonly used in solid mechanics and in most works of growth mechanics, the spatial description has certain advantages in specifying the growth rule, which is a key issue in developing a theory for surface growth. Then we consider the case where the growth is taking place only in a region bounded by two adjacent moving surfaces. By passing to the limit as the thickness of this region tends to zero, we obtain a theory for surface growth. In this limiting process, the spatial description is carried through, resulting in discontinuous velocity field and singular growth rate field. On the other hand, the material description breaks down in the limiting process, as the oneto- one correspondence of the material points in the reference and current configurations ceases to exist. However, we show that a change of variables can be made in the material description by using the time when a material point is leaving the moving surface as an independent variable. The resulting theory agrees in spirit with that of Skalak, et al. [1], where the time when a material is created is used as an independent variable.

Such development also leads to a unified theory for both surface and volumetric growths by using the spatial description. In this theory, all state variables, such as velocity, density, stress, growth rate, etc., are generalized functions (distributions) of time and position in space. While the volumetric growths are characterized by smooth functions over the space, the surface growths are characterized by discontinuous and singular functions on some moving surface in the space.

References

[1] R. Skalak, G. Dasgupta, M. Moss, E. Otten, P. Dullemeijer, and H. Vilmann, Analytical description of growth. *J. Theor. Biol.* **94** (1982) 555–577.