

# MATHEMATICS of FAIRNESS

**HKMS Texts in General Education**

Volume 1

SUNG NOK CHIU  
LEE VAN LING



# Preface

The arrival of universal suffrage in Hong Kong, currently the most popular political topic in town, remains surrounded by uncertainty. We still do not know when-or whether-the dream of universal suffrage will come true, as the Government claims that it still has a significant number of issues to consider, the most important of which is which election method to adopt. Hong Kong, a so-called democratic city, has never had a truly democratic election. Hong Kong citizens do not have a single, agreed-upon concept of the best election system, and most of us are politically apathetic. Political issues are different from cake-cutting problems in mathematics, but there are many similarities between the two. To encourage a passionate interest in problems of fairness, this book presents students with a variety of relatively simple examples that are familiar from our everyday lives. These examples are intended to consolidate students' mathematical knowledge and provide them with a foundation for tackling such real-life problems as Legislative Council (LegCo) elections. Real-life, local examples not only offer students a sense of familiarity, and render all of the techniques they will acquire in this course applicable to the Hong Kong situation, but equip them with logical-thinking and problem-solving skills.

## **Objectives**

This textbook is intended for use in university-level general education courses. It is written in a clear and concise manner, with numerous real-life examples offered in each chapter. There are six chapters in total, each of which introduces one major fairness concept. The book is not intended to teach students calculation processes, but rather the logical processes that are involved in obtaining answers to problems we encounter in daily life.

The book begins with an introduction to the way in which statistical information

can be used to interpret and affect election phenomena and other everyday issues. Three very basic, but sometimes confusing, statistical terms—mean, mode and median—are employed to analyze a variety of Hong Kong-specific examples. We then cover more advanced decision-making concepts. Students will learn about a number of election systems and address the problem of finding the best decision-making procedures. Our ultimate goal is to identify an election system that can turn individual preferences for different candidates into a single choice made by an entire group. Moreover, a variety of possible methods of electing our future Chief Executive are investigated (although we cannot promise you that he or she will ever be elected through universal suffrage). In the process, we consider the pros and cons of each method, including their vulnerability to manipulation. After equipping you with knowledge of different election systems, we then introduce you to the weighted voting system, in which voters may be treated unequally. This may be a novel concept for you, but you will soon see that, in many cases, the only way to treat voters fairly is to assign them unequal degrees of power. The book next turns to an investigation of a number of simplified, but real-life cases, such as the distribution of seats in LegCo. Finally, we address the mathematically well-known cake-cutting problem, which is formally known as the fairness problem. When the demands or desires of one party are in conflict with those of another, how can we meet or satisfy those demands or desires in such a way that both parties believe that they have received their fair share? You will find the answer—and those to many other fairness-related questions—in this book.

### **To the students**

In response to the new 3-3-4 curriculum, the academic structure will be more focused than ever on all-rounded personal development. The need for language ability and mathematical skill has risen significantly and therefore new courses have been introduced to meet these requirements. To streamline the teaching and learning processes, we have created a custom series of books. *Mathematics of Fairness* assists students in developing the most vital concept in a democratic city—concept of fairness. Grasping the relationship between mathematics and fairness demands a wider scope and deeper understanding of mathematical modeling. This book demonstrates how mathematics can help in searching for procedures that allow for fair and equitable resolutions to conflicts.

Topics that are covered include algorithms for envy-free sharing, fair division pro-

cedures for labor-management negotiations, the relevance and theory behind mathematics in voting systems, calculations of voting power and its implications to society, and methods of apportionment concerning seat allocation within elections. This book attempts to capture local examples from real-life situations in Hong Kong in order to ensure that the material is applicable to its readers. It is our hope that the concepts and skills portrayed in this book will help readers identify, model, and solve social problems that they may encounter. Under proper guidance, it is not out of your reach. Enjoy your learning!

### **Acknowledgements**

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*Hong Kong, November 2010*

*Leevan Ling*

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