The Riemann-Hilbert Method

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In its original setting, the Riemann-Hilbert problem is the question of surjectivity of the monodromy map in the theory of Fuchsian systems, and it was included by Hilbert in his famous list as problem number twenty-one. Subsequent developments put the Riemann-Hilbert problem into the context of analytic factorization of matrix-valued functions and brought to the area the methods of singular integral equations (Plemelj, 1908) and holomorphic vector bundles (Röhrl. 1957). This resulted eventually in a (negative) solution, due to Bolibruch (1989) of the Riemann-Hilbert problem in its original setting and to a number of deep results (Bolibruch, Kostov) concerning a thorough analysis of relevant solvability conditions.

Simultaneously, and to a great extent independently of the solution of the Riemann-Hilbert problem *per se*, a powerful analytic apparatus - *the Riemann-Hilbert method* - was developed for solving a vast variety of problems in pure and applied mathematics. A classical example of the use of analytic factorization techniques is the Wiener-Hopf method in linear elasticity, hydrodynamics, and diffraction.

Another array of problems that have fallen under the Riemann-Hilbert formalism over the last twenty - twenty five years came from modern theory of integrable systems. In this new area, the Riemann-Hilbert approach exploits ideas which go beyond both the usual Wiener-Hopf scheme and the theory of singular integral equations, and they have their roots in the inverse scattering method of soliton theory and in the theory of isomonodromy deformations. The main “beneficiary” of this, latest version of the Riemann-Hilbert method, is the global asymptotic analysis of nonlinear systems. Indeed, many long-standing asymptotic problems in the diverse areas of pure and applied math have been recently solved with the help of the Riemann-Hilbert technique.

In this talk a general overview of the Riemann-Hilbert method will be given. The most recent applications of the Riemann-Hilbert approach to asymptotic problems arising in the theory of matrix models, orthogonal polynomials, and statistical mechanics will be outlined. The talk is based on the works of many authors spanned over a number of years.