Introduction to Zonotopal Algebra

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The most common methodology for constructing multivariate splines is via their definition as volume functions. One begins with a linear surjection

$$X : \mathbb{R}^N \to \mathbb{R}^n,$$

and restricts this map to a polyhedron $\mathbb{Z} \subset \mathbb{R}^N$. In the theory of **box splines**, $\mathbb{Z} = [0,1]^N$. Two geometries underscore box spline theory: that of *zonotopes*, and the dual geometry of *hyperplane arrangements*. The geometries are associated with dual algebraic structures, and results in a seamless cohesion of the geometry, the algebra, the spline functions and combinatorial properties of X.

Attempts to extend the aforementioned constructions beyond the original setup of box spline theory began in the mid 90's and reached their successful completion in [3] where a three-layer theory that was coined there *zonotopal algebra* is introduced. Box spline theory is the middle *central* layer and two novel constructions, external and internal, over the same geometries were newly introduced.

Zonotopal Algebra rests on two fundamental principles that interweave the geometry, the combinatorics and the algebra. The first is the fact that the algebraic structures can be derived from the geometry (via a non-linear procedure known as "the least map", [1]), and the second is that the statistics of the algebraic structures (e.g., the Hilbert series of various polynomial ideals) are combinatorial, i.e., computable using a simple discrete algorithm known as "the valuation function", [2]. There are several recent extensions (e.g., [4], [5]) of Zonotopal Algebra (beyond the 3-layer theory) that retain the above principles.

I will not be able in this short talk to review seriously any aspect of Zonotopal Algebra. Instead, I will provide a gentle, friendly discussion that illustrates the way the theory connects splines, geometry, algebra and combinatorics, as well as the nature of the two principles that are mentioned above. No knowledge of box spline theory is assumed.

The talk is based in part on joint work with Nan Li (MIT), Olga Holtz (Berkeley) and Zhiqiang Xu (Chinese Academy of Sciences, Beijing).

References

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