Gabor Frames and Totally Positive Functions

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We investigate properties of Gabor frames 
\[ G(g, \alpha, \beta) = \{ M_{l\beta}T_{k\alpha}g : k, l \in \mathbb{Z} \} \]
where \( g \in L^2(\mathbb{R}) \) and \( M_{l\beta}T_{k\alpha}g(x) = e^{2\pi i \frac{l}{\beta}x}g(x - k\alpha) \). In collaboration with K. Gröchenig, we proved that \( G(g, \alpha, \beta) \) is a frame if \( \alpha \beta < 1 \) and \( g \) is a totally positive function of finite type, i.e., its Fourier transform is 
\[ \hat{g}(\omega) = C \prod_{j=1}^{m}(1 + 2\pi i \delta_j \omega)^{-1}, \quad \delta_j \in \mathbb{R} \setminus \{0\}. \]

With T. Kloos from Dortmund, we found new properties of these frames:

1. The Zak transform
\[ Zg(x, \omega) = \sum_{k \in \mathbb{Z}} g(x + k)e^{-2\pi ik\omega} \]
has exactly one zero in \([0, 1)^2\). This result follows from the close connection of \( g \) with suitably normalized exponential B-splines.

2. The lower frame bound of \( G(g, 1, \beta) \), where \( g(x) = e^{-|x|} \) is the totally positive function with \( m = 2 \) and \( \delta_{1,2} = \pm 1 \), satisfies
\[ A \geq c \left( \frac{1}{\beta} - 1 \right), \quad 0 < \beta < 1. \]

Similar estimates for the lower frame bound were shown to exist for the Gaussian by Gröchenig and Lyubarskii. We also propose a method from approximation theory (de Boor’s lemma for norms of inverses of totally positive matrices) to extend our second result to totally positive functions of finite order.