A Meshless Computational Method for Solving Inverse Heat Conduction Problem

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Abstract

In this paper, a new meshless numerical scheme for solving inverse heat conduction problem is proposed. The numerical scheme is developed by using the fundamental solution of heat equation as basis function and treating the entire space-time domain in a global sense. The standard Tikhonov regularization technique and L-curve method are adopted for solving the resultant ill-conditioned linear system of equations. The approach is readily extendable to solve high-dimensional problems under irregular domain.

1 Introduction

Consider a linear inverse heat conduction problem (IHCP) where the boundary heat flux has to be determined from the temperature measurements in the interior or on the boundary of the domain. A number of solution techniques have been proposed for a one-dimensional IHCP [3] [4]. It is understood that, in practical situation, only scattered data can be obtained [2]. Two special cases of these IHCPs are formulated in the following:

**Problem A:** Determine the temperature and heat flux on surface $x = 0$ from the following equation and initial-boundary conditions:

\[ u_t - a^2 u_{xx} = 0, \quad 0 < x < 1, \quad 0 < t < t_{\text{max}}, \quad (1) \]
\[ u(x, 0) = \phi(x), \quad 0 \leq x \leq 1, \quad (2) \]
\[ u(1, t) = g(t), \quad 0 \leq t \leq t_{\text{max}}, \quad (3) \]
\[ \frac{\partial u}{\partial x}(1, t) = h(t), \quad 0 \leq t \leq t_{\text{max}}, \quad (4) \]

where \( u(x, t) \) is the temperature distribution and \( t_{\text{max}} \) represents the maximum time of interest for the time evolution of the problem. This problem is well known for being highly ill-posed.

In practical application, if discrete temperature data are specified at discrete values of time \( t_i^{(1)}, i = 1, 2, \ldots, m \), these values of temperature are denoted by \( g_i, i = 1, 2, \ldots, m \) and the boundary condition (3) will be replaced as follow:

\[ u(1, t_i^{(1)}) = g_i, \quad i = 1, 2, \ldots, m, \quad (5) \]

where \( t_1^{(1)}, t_2^{(1)}, \ldots, t_m^{(1)} \) denote the discrete values of times in the internal \((0, t_{\text{max}}]\).

Problem B: Consider the heat equation given in (1) and the initial condition prescribed in (2) plus either the boundary condition specified in (3) or the discrete setting in (5). Determine the temperature \( u(0, t) \) and heat flux \( \frac{\partial u}{\partial x}(0, t) \) from the temperature measurements at an internal point \( x^* \in (0, 1) \), i.e., either

\[ u(x^*, t) = h(t), \quad 0 < t \leq t_{\text{max}}, \quad (6) \]

or at discrete values of time

\[ u(x^*, t_i^*) = h_i, \quad i = 1, 2, \ldots, l, \quad (7) \]

where \( \{t_1^*, t_2^*, \ldots, t_l^*\} \subset (0, t_{\text{max}}] \).

In the following, we present a new approach to solve both Problem A and Problem B respectively. We propose in this paper an alternative method for solving some inverse heat conduction problems. It is also worthwhile to mention that in Frankel et al. [2], an unified space-and-time treatment of the inverse heat conduction problem by using the Chebyshev polynomials as basis functions has been investigated. The proposed method in this paper is to apply the fundamental solution of heat equation as basis functions. This approach is different from most existing numerical algorithms in solving dynamical problems where the finite difference quotient will be used to discretize the time variable. For stable computation, the standard Tikhonov regularization technique and L-curve method [13]–[16] are adopted to solve the resultant ill-conditioned linear systems.

2 Fundamental solution method

The fundamental solution of equation (1) is given as

\[ F(x, t) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} H(t), \quad (8) \]
where \( H(t) \) is the Heaviside function.

Assume that \( T > t_{\text{max}} \) is a constant. We then have

\[
\phi(x, t) = F(x, t + T),
\]

which is a general solution of equation (1) on the domain \([0, 1] \times [0, t_{\text{max}}]\).

Now, assume that the known data are collected at the collocation points
\((x_j, t_j) = (\frac{j-1}{n-1}, 0), j = 1, 2, \ldots, n\) satisfying the initial condition (2) on the
line \( t = 0 \) and the points \((x_j, t_j) = (1, t^{(1)}_{j-n}), j = n + 1, n + 2, \ldots, n + m\) on
the boundary \( x = 1 \) satisfying the boundary condition (5), where \( \{t^{(1)}_{i}\}_{i=1}^{m} \)
is measurement times given in (5). For Problem A: in order to use the heat
flux data (4), we choose the collocation points \((x_j, t_j) = (1, t^{(2)}_{j-n-m}), j = n + m + 1, n + m + 2, \ldots, n + m + l\) on
the boundary \( x = 1 \), where \( \{t^{(2)}_{i}\}_{i=1}^{l} \subset (0, t_{\text{max}}] \) and \( \{t^{(1)}_{i}\}_{i=1}^{m} \cap \{t^{(2)}_{i}\}_{i=1}^{l} = \emptyset \). For Problem B: the measurement
condition (7) substitutes the heat flux condition (4) and in this case we
choose the collocation points on \( x = x^* \) at the given discrete times \((x_j, t_j) = (x^*, t^{(1)}_{j-n-m}), j = n + m + 1, n + m + 2, \ldots, n + m + l, \) where \( x^* \) and \( \{t^{(1)}_{i}\}_{i=1}^{m} \) arise from equation (7).

In both Problems A and B, all the collocation points \( \{(x_j, t_j)\}_{j=1}^{n+m+l} \) are
pairwise distinct in the two dimensional space \([0, 1] \times [0, t_{\text{max}}]\).

Following the idea of the method of fundamental solution for solving elliptic
equation in [11, 10], we assume that an approximation to the solution of
Problem A or Problem B can be expanded by the following basis functions:

\[
u^*(x, t) = \sum_{j=1}^{n+m+l} \lambda_j \phi(x - x_j, t - t_j),
\]

where the basis function \( \phi(x, t) \) is given by equation (9) and \( \lambda_j \) are unknown
coefficients.

For this choice of basis functions \( \phi \), the approximated solution \( u^* \) already
satisfies the heat equation (1). The resultant linear system of equations for
the unknown coefficients \( \lambda_j \) for Problem A and Problem B can then be
obtained by the following simple collocation.

Problem A: Collocating (10) into (2), (5) and (4), we obtain

\[
u^*(x_i, t_i) = \sum_{j=1}^{n+m+l} \lambda_j \phi(x_i - x_j, t_i - t_j),
\]

and

\[
\frac{\partial u^*}{\partial x}(x_i, t_i) = \sum_{j=1}^{n+m+l} \lambda_j \frac{\partial \phi}{\partial x}(x_i - x_j, t_i - t_j),
\]
= h(t_i), \quad i = n + m + 1, n + m + 2, \cdots, n + m + l.

In matrix form, the values of the unknown coefficients \( \lambda_j \) can be obtained from solving the following matrix equation:

\[
A \lambda = b, \tag{11}
\]

with

\[
A = \begin{pmatrix}
\phi(x_i - x_j, t_i - t_j) \\
\frac{\partial}{\partial x}(x_k - x_j, t_k - t_j)
\end{pmatrix}, \tag{12}
\]

where \( i = 1, \cdots, n+m, \quad k = n+m+1, \cdots, n+m+l, \quad j = 1, \cdots, n+m+l, \)

and

\[
b = \begin{pmatrix}
\varphi(x_i) \\
g_{(i-n)} \\
h_{(t_i)}
\end{pmatrix}_{n+m+l,1}. \tag{13}
\]

Problem B: Similarly we can obtain the linear system (11) for Problem B by collocating (10) on (2), (5), and (7) whereas

\[
A = (\phi(x_i - x_j, t_i - t_j))_{m+n+l,m+n+l}, \tag{14}
\]

and

\[
b = \begin{pmatrix}
\varphi(x_i) \\
g_j \\
h_k
\end{pmatrix}_{i=1,\cdots,n, \quad j=1,\cdots,m, \quad k=1,\cdots,l}. \tag{15}
\]

In the next section, the linear system (11) will be solved to give an approximated solution for both Problem A and Problem B by substituting the coefficient vector \( \lambda \) into equation (10).

3 Regularization methods

In the last section, we have obtained the linear system (11) for the IHCP as Problem A or Problem B. Since the original IHCP is ill-posed, the ill-conditioning of the matrix \( A \) in equation (11) still persists. In other words, most standard numerical methods cannot achieve good accuracy in solving the matrix equation (11) due to the bad condition number of the matrix \( A \). In fact, the condition number of matrix \( A \) increases dramatically with respect to the total number of collocation points. Several regularization methods have been developed for solving these kinds of ill-conditional problems [13]–[16]. In our computation we adopt the Tikhonov regularization [12] to solve the matrix equation (11). The Tikhonov regularized solution
\( \lambda \) for equation (11) is defined as the solution to the following least square problem:
\[
\min_{\lambda} \{ \| A\lambda - b \|^2 + \alpha^2 \| \lambda \|^2 \}, \quad (16)
\]
where \( \| \cdot \| \) denotes the usual Euclidean norm and \( \alpha \) is called the regularization parameter.

The determination of a suitable value of the regularization parameter \( \alpha \) is crucial and is still under intensive research (refer [12]). In our computation we use the L-curve method to determine a suitable value of \( \alpha \). The L-curve method was firstly applied by Lawson and Hanson [18] and more recently by Hansen and O’Leary [13] to investigate the properties of regularized systems under different values of the regularization parameter \( \alpha \). The L-curve method is sketched in the following:

Define the curve
\[
L = \{ \log(\| A\lambda \|), \log(\| A\lambda - b \|), \quad \alpha > 0 \}. \quad (17)
\]
The curve is known as L-curve and a suitable regularization parameter \( \alpha \) is one that corresponds to a regularized solution near the “corner” of the L-curve [14]–[16].

In our computation, we used the MatLab code developed by Hansen [17] for solving the discrete ill-conditioned system (11). Denote the regularized solution of (11) by \( \lambda^\alpha \). The approximated solution \( u^\alpha \) for Problem A or Problem B is then given as
\[
u^\alpha(x, t) = \sum_{j=1}^{n+m+l} \lambda_j^\alpha \phi(x - x_j, t - t_j). \quad (18)
\]
The temperature and heat flux at surface \( x = 0 \) can be easily computed.

There is still no convergence and error estimates proof available. The numerical results in the following indicate that the proposed scheme is feasible and efficient.

4 Numerical results

For simplicity, we assume that the heat conduction coefficient \( a = 1 \) and \( t_{max} = 1 \).

For numerical error estimation on the approximation, we choose 101 x 101 test points on the domain \([0, 1] \times [0, 1]\) and compute the root mean square error by the following formula
\[
RMS(u) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (u_i - u_i^\*)^2}, \quad (19)
\]
where \( N \) is the number of test points on the domain \([0, 1] \times [0, 1]\), \( u_i \) and \( u_i^\* \) are respectively the exact and approximated temperature value at these
test points. The $RMS\left(\frac{\partial u}{\partial x}\right)$ is the RMS error for the heat flux on the whole domain. Similarly, we denote $RMS(u(0,t))$ and $RMS\left(\frac{\partial u}{\partial x}(0,t)\right)$ as the RMS errors for the surface temperature and heat flux on $x = 0$ respectively.

In the cases when the measurement data include some random noises, we use noisy data $\tilde{g}_i = g_i + \sigma \text{rand}(i)$, where $g_i$ is the exact data and $\text{rand}(i)$ is a random number between $[-1, 1]$ and the value of $\sigma$ indicates the error level.

Example 1: The exact solution of Problem A is chosen as

$$u(x,t) = e^{-4t}(\sin(2x) + \cos(2x)),$$

where the boundary data $g_i$, $h(t)$ and initial condition $\varphi(x)$ can be induced from the solution. The values of the temperature on $x = 1$ are used with the exact data $g_i$ and noisy data $\tilde{g}_i$. The numerical results obtained by our method are presented in Figures 1 and 2.

![Figure 1: The error distribution for $u(x,t)$ and $\frac{\partial u}{\partial x}(x,t)$ in the case of zero noise data.](image)

Example 2: Taking an exact solution of Problem B as equation (20). The comparisons between the exact solution and the approximation are given in Figure 3 and Figure 4 on their values of temperature and heat flux on the boundary $x = 0$.

Example 3: To further explore the application of the proposed method on solving IHCP, we consider the classic Beck’s problem [2] where triangular surface heat flux is imposed at $x = 0$. Assume that an exact heat flux on
Figure 2: The boundary temperature $u(0, t)$ and heat flux $\frac{\partial u}{\partial x}(0, t)$ for discrete noisy data $\tilde{g}_i$, $\tilde{g}_i = g_i + \sigma \text{rand}(i), \sigma = 0.01$. The solid line represents the exact solution and the dotted line represents the approximation.

Figure 3: The boundary temperature $u(0, t)$ and heat flux $\frac{\partial u}{\partial x}(0, t)$ for zero noise data in Problem B. Take $x^* = 0.9$. The solid line represents the exact solution and the dotted line represents the approximation.

$x = 0$ is given as

$$f(t) = -\frac{\partial u}{\partial x}(0, t) = \begin{cases} 0, & t \in [0, 0.25] \cup [0.75, 1], \\ \frac{t-0.25}{0.25}, & t \in [0.25, 0.5], \\ \frac{t-0.75}{0.25}, & t \in [0.5, 0.75]. \end{cases}$$

(21)

Let $h(t)$ in (4) be zero and the initial condition $\varphi$ in (2) is also zero. By using the standard technique in the theory of partial differential equations,
parameter T=2.1

Figure 4: The boundary temperature $u(0,t)$ and heat flux $\frac{\partial u}{\partial x}(0,t)$ for noisy data $\tilde{h}_i$ in Problem B. $\tilde{h}_i = h_i + \sigma \text{rand}(i)$, $\sigma = 0.01$. The solid line represents the exact solution and the dotted line represents the approximation.

the boundary condition $g(t)$ in (3) of Problem A is given as

$$g(t) = u(1,t) = \int_0^t f(\tau)d\tau + \sum_{n=1}^{\infty} 2(-1)^n \int_0^t f(\tau)e^{-(n^2/2)(t-\tau)d\tau}.$$  \hspace{1cm} (22)

From the data $g(t)$, we can then reconstruct the triangular surface heat flux (21). The predicted surface heat flux and the exact one (21) is displayed in Figures 5 and 6.

5 Conclusion

The universal approach for solving transient phenomena, directly or inversely, involves a time-marching procedure, i.e. taking one step in time and solving the problem in the spatial domain. In this paper, by inverting the resultant matrix, an approximated solution on the entire domain is obtained. This ensures that the time consuming iterative solving process is effectively formulated. We believe that the proposed RBFs-MFS approach provides an efficient global time-space scheme for solving these kinds of inverse heat conduction problems.

References

Figure 5: The boundary temperature $u(0, t)$ and heat flux $\frac{\partial u}{\partial x}(0, t)$ in the case of zero noise data $g(t)$ for Example 3. The solid line represents the exact solution and the dashed line represents the approximation.

Figure 6: The boundary temperature $u(0, t)$ and heat flux $\frac{\partial u}{\partial x}(0, t)$ with noisy data $\tilde{g}(t)$ for Example 3. $\tilde{g}(t) = g(t) + \sigma rand(t), \sigma = 0.01$ The solid line represents the exact solution and the dashed line represents the approximation.


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