Boundary knot method for 2D and 3D Helmholtz and convection-diffusion problems under complicated geometry

Y. C. Hon*
Department of Mathematics, City University of Hong Kong, Hong Kong SAR, China

W. Chen
Department of Informatics, University of Oslo, Oslo, Norway

ABSTRACT

The boundary knot method (BKM) has recently been developed as an inherently meshless, integration-free, boundary-type collocation technique for the numerical discretization of general partial differential equation systems. Unlike the method of fundamental solutions, the use of non-singular radial basis functions in the BKM avoids the unnecessary requirement of constructing a controversial artificial boundary outside the physical domain. The purpose of this paper is to extend the BKM to solve 2D and 3D Helmholtz and convection-diffusion problems under rather complicated irregular geometry. Numerical experiments validate that the BKM can produce highly accurate solutions using only a small number of nodes. For inhomogeneous cases, some inner nodes are needed to guarantee accuracy and stability. The completeness, stability and convergence of the BKM are also numerically illustrated.

KEY WORDS: boundary knot method, radial basis function, non-singular general solution, method of fundamental solutions, boundary elements, meshless.

*Corresponding author. This work was also partially supported by a strategic research grant number 7001051 of the City University of Hong Kong.
1. INTRODUCTION

The recent advances on meshless numerical techniques have become increasingly popular since the construction of a mesh in the standard finite element and boundary element methods is not a trivial work especially for nonlinear, moving boundary and higher-dimensional problems [1 - 3]. Among these meshless techniques, the local boundary integral equation (MLBIE) method [4], the boundary node method (BNM) [5], and the method of fundamental solutions (MFS) [3, 6] are typically meshless boundary-type numerical schemes. The essence of the MLBIE and BNM is basically a combination of the moving least square (MLS) technique with the boundary element scheme, whereas the MFS employs a collocation scheme by using the radial basis function (RBF) that requires a construction of an artificial boundary outside the physical domain to avoid the singularity of the fundamental solution. The MLBIE and BNM generally involve singular integration and hence are mathematically more complicated in comparing with the commonly used finite element method (FEM). In addition, their low order approximations also reduce the efficiency of the MLBIE and BNM [3]. In fact, since the BNM still requires meshes in its numerical integration, it is not a truly meshless scheme compared with the MLS-based meshless FEM [4, 7]. On the other hand, the MFS possesses integration-free, spectral convergence, and easy-to-use merits [3,6]. However, the requirement of an arbitrary fictitious boundary hinders the practical applicability of the MFS [3, 8] and its resulting interpolation matrix often causes severe ill-conditioning problem [9].

Chen and Tanaka [10, 11] recently developed a boundary knot method (BKM) as an alternative boundary-type meshless RBF collocation scheme. The BKM is basically a combination of the Trefftz-type technique [12] with the RBF, non-singular general solution, and dual reciprocity method (DRM). The RBF is employed in the BKM to approximate the inhomogeneous terms via the DRM, whereas the non-singular general solution of the partial differential operator leads to a boundary-only RBF formulation for the homogeneous solution. It is worth stressing that the BKM eliminates the inherent inefficiency of the MFS due to the use of an auxiliary surface outside the physical domain. Furthermore, the use of the non-singular general solution in the BKM instead of singular fundamental solution in the MFS avoids the unnecessary construction of a controversial fictitious boundary. It is noted here that
the BKM is theoretically applicable to general linear and nonlinear inhomogeneous partial differential equation [11] as compared with the dual reciprocity boundary element method (DRBEM) [13,14] and the MFS [3]. Moreover, due to the use of the radial basis function, the BKM is essentially meshless for solving any dimensional problems of various types of partial differential equations [10]. In addition, the method is mathematically simple and easy to implement.

Due to the recent development of the BKM, the method has so far only been applied to solve linear and nonlinear Dirichlet problems defined on a smooth 2D elliptic domain [10, 11, 15]. This paper aims to extend the BKM to solve both 2D and 3D Helmholtz and convection-diffusion problems under rather complicated domains with irregular boundaries. To the knowledge of the author, this will be a first time the BKM is shown to be effective in solving 3D problems. Numerical experiments are very encouraging in terms of efficiency, accuracy, stability, and simplicity. The stability and convergence rate of the method are also numerically illustrated. Some open issues are raised in the conclusion section.

2. BOUNDARY KNOT METHOD

The BKM can be illustrated by a two-step numerical approach [10, 11]. Firstly, the DRM and RBF are employed to evaluate the particular solution of the problem, and secondly, its homogeneous solution is calculated by using non-singular general solution formulation. Without loss of generality, consider the following multi-dimensional convection-diffusion equation

\[ D(x)\nabla^2 u(x) - v(x) \cdot \nabla u(x) - ku(x) = f(x), \quad (1) \]

\[ u(x) = D(x), \quad x \in S_u, \quad (2a) \]

\[ \frac{\partial u(x)}{\partial n} = N(x), \quad x \in S_T, \quad (2b) \]
where \( v \) denotes a velocity vector, \( D \) is the diffusivity coefficient, \( k \) represents the reaction coefficient, \( S_u \) and \( S_T \) denote the boundaries with Dirichlet and Neumann conditions respectively, \( x \) is the multi-dimensional independent variable, and \( n \) is the unit outward normal. The solution \( u \) can be expressed as

\[
u = u_h + u_p,
\]

where \( u_h \) and \( u_p \) are the homogeneous and particular solutions of the problem respectively. In other words, the particular solution \( u_p \) satisfies

\[
D(x)\nabla^2 u_p - v(x) \cdot \nabla u_p - ku_p = f(x),
\]

but does not necessarily satisfy the boundary conditions. The homogeneous solution \( u_h \) satisfies

\[
D\nabla^2 u_h - v \cdot \nabla u_h - ku_h = 0,
\]

\[
u_h(x) = D(x) - u_p, \quad x \in S_u,
\]

\[
\frac{\partial u_h(x)}{\partial n} = N(x) - \frac{\partial u_p(x)}{\partial n}, \quad x \in S_T.
\]

The computation for these particular and homogeneous solutions by using a two-step BKM scheme will be introduced in the following sections.

2.1. Particular solution by the DRM and RBF

Based on the DRM and RBF [13, 14], the inhomogeneous term \( f(x) \) of equation (4) can be approximated by
where $\alpha_k$ are the unknown coefficients to be determined, $N$ and $L$ represent respectively the total numbers of knots on the domain and the boundary respectively, $r_k = \|x - x_k\|$ denotes the Euclidean distance between each $x$ and knot $x_k$, and $\phi$ is called RBF. The polynomial term $\varphi$ is added to guarantee the non-singularity of the resultant RBF interpolation matrix for the case when the chosen RBF is only conditionally positive definite. In general, $\varphi$ is chosen to be a series of linear polynomials. Refer [16,17] for details.

From equation (7) we can uniquely determine each $\alpha_k$ by

$$\alpha = A^{-1}_x f(x),$$

where $A_x$ is a non-singular RBF interpolation matrix. From equation (8), the particular solution $u_p$ at any points can be obtained by summing all the localized particular solutions

$$u_p = \sum_{k=1}^{N+L} \alpha_k \psi(r_k),$$

where each $\psi(r_k)$ satisfies the equation

$$\phi(r) = \nabla^2 \psi(r).$$

In general, it is not a trivial task to integrate equation (10) to obtain the approximated particular solution $\phi$ for arbitrary RBF $\phi[3]$. Recently, Muleskov et al. [18] derived an analytic formula to compute the approximated particular solutions for Helmholtz operator by using the polyharmonic splines as RBF $\phi$. Unfortunately, in the case of convection-diffusion operator, such analytical approximation is still not available. In our practical computation, we apply a reverse procedure for the particular solution $\psi(r_k)$ of equation (1). In this reverse procedure, the approximated particular solution
\( \phi \) is specified beforehand, and then the corresponding RBF \( \phi \) is evaluated by simply substituting the specified \( \phi \) into equation (10). This scheme works well for various types of problems in the dual reciprocity BEM [19,20].

2.2. Boundary formulation with non-singular general solution

The homogeneous solution \( u_h \) from equations (5) and (6a,b) can be obtained by using various boundary-type numerical techniques [3, 14]. In the BKM, we employ instead a non-singular general solution. For illustration, in the case of the convection-diffusion operator, the non-singular general solution is as follow:

\[
\tilde{\sigma}_n(r) = \frac{1}{2\pi} \left( \frac{\mu}{2\pi r} \right)^{(n/2)-1} e^{-\frac{vr}{2D}} I_{(n/2)-1}(\mu r), \quad n \geq 2. \tag{11}
\]

For comparison, the corresponding fundamental solution is

\[
u^*_n(r) = \frac{1}{2\pi} \left( \frac{\mu}{2\pi r} \right)^{(n/2)-1} e^{-\frac{vr}{2D}} K_{(n/2)-1}(\mu r), \quad n \geq 2, \tag{12}
\]

where \( n \) is the dimension of the problem; \( I \) and \( K \) are respectively the modified Bessel functions of the first kind and the second kind; and

\[
\mu = \left[ \left( \frac{v}{2D} \right)^2 + \frac{k}{D} \right]^{\frac{1}{2}}. \tag{13}
\]

It is easy to use any symbolic software package such as Maple to verify that both functions given in equations (11) and (12) satisfy the following homogeneous convection-diffusion equation

\[
D(x) \nabla^2 \phi + v(x) \cdot \nabla \phi - k \phi = 0. \tag{14}
\]
It is also noted that the signs of the velocity terms in equations (1) and (14) are opposite since the operator is not self-adjoint. The only difference between the non-singular general solution and the singular fundamental solution is that the former uses the modified Bessel function of the first kind whereas the latter uses instead the modified Bessel function of the second kind. Furthermore, the fundamental solution (12) has a singularity at the origin.

Let \( \{x_j\}_{j=1}^L \) represent a set of knots on the physical boundary. The homogeneous solution \( u_h(x) \) of equation (5) can be approximated by the following series

\[
v(x) = \sum_{j=1}^L \beta_j \sigma_n(r_j),
\]

where \( r_j = \|x - x_j\|, \) \( L \) is the total number of boundary knots, \( \beta_j \) are unknown coefficients to be determined. Collocating equation (6a,b) in terms of the series (15) gives

\[
\sum_{j=1}^L \beta_j \sigma_n(r_{ij}) = D(x_i) - u_p(x_i), \tag{16a}
\]

\[
\sum_{j=1}^L \beta_j \frac{\partial \sigma_n(r_{jm})}{\partial n} = N(x_m) \frac{\partial u_p(x_m)}{\partial n}, \tag{16b}
\]

where \( i \) and \( m \) indicate the Dirichlet and Neumann boundary response knots respectively. In the case if the inhomogeneous term in equation (1) involves the dependent unknown \( u \), say \( f(x, u) \) is given instead of \( f(x) \), we need to constitute a set of supplementary equations for the unknowns at the inner nodes as follow:

\[
\sum_{j=1}^L \beta_j \sigma_n(r_{lj}) = u_i - u_p(x_i), \quad l = 1, \ldots, N, \tag{17}
\]

where \( l \) denotes the index of each internal response knot and \( N \) is the total number of interior points. We have then obtained a total of \( N + L \) simultaneous algebraic
equations for the unknown coefficients $\alpha_k$ of equation (9). By solving the simultaneous equations (16a, b) and (17), we obtain the values of the undetermined coefficients $\beta_j$ and the solutions values at the $N$ internal nodes. It is remained to calculate the solution $u$ value at any inner knot by

$$u(x) = u_b(x) + u_p(x) = \sum_{j=1}^{L} \beta_j \sigma_a(r_j) + \sum_{k=1}^{N+L} \alpha_k \psi(r_k),$$  \hspace{1cm} (18)

Unlike the MFS, all boundary collocation knots $x_j$ in the BKM are placed only on the physical boundary and can be treated as either source or response points. From [11, 15], it is straightforward to extend this convection-diffusion operator to the other differential operators such as the Helmholtz, modified Helmholtz, and biharmonic.

As was pointed out in [3,21], the DRM with the RBF is a meshless technique for evaluating particular solution of general PDEs, while the non-singular general solution formulation for homogeneous solution is also an essentially meshless RBF boundary-type methodology. The proposed two-step BKM scheme, however, constructs a truly meshless numerical discretization technique for general higher-dimensional problems.

3. APPLICATIONS AND DISCUSSIONS

In this section the applications of the BKM to solve both 2D and 3D Helmholtz and convection-diffusion problems will be illustrated. As was mentioned earlier, the main purpose of this paper is to verify the applicability of the BKM to solve PDE problems with arbitrarily irregular boundary. In the BKM, the inhomogeneous term is approximated by using the DRM and RBF which is similar to DRBEM and MFS. Therefore, the major emphasis on the BKM solution is on its applicability to finding the solutions of the corresponding homogeneous problems. In this paper, an inhomogeneous case is given to indicate that some internal nodes are necessary for the DRM evaluation of the particular solution in the BKM solution of inhomogeneous problems. As shown in the previous section, the application of the BKM to more
complicated inhomogeneous problems is straightforward [11,15]. Furthermore, to the knowledge of the authors, this paper gives the first attempt to apply the BKM to solve 3D problems.

Unless otherwise specified, all 2D tested cases have the same configuration of irregular geometry with Neumann and Dirichlet boundary conditions at \( x = 0 \) and \( y = 0 \) respectively as shown in Figure 1. It is also noted that this configuration involves corners, sharp notches, and interior elliptical and rectangular cut-outs. Such a configuration with these interior and exterior boundary shapes are deliberately designed to verify the robustness of the BKM in solving arbitrary complicated geometric problems. For the 3D case, the configuration is given in Figure 2 which is a 3D cube with all sides of equal length. Both the Helmholtz and convection-diffusion problems with this 3D cube geometry are tested to verify the simplicity, efficiency, and accuracy of the BKM for solving higher-dimensional problems. The experimental problems were taken from [14] with some modifications.

All tested results are displayed in Tables 1-5. The relative error of the BKM solution, which is defined to be the ratio of the absolute approximation error to the absolute value of the actual analytical solution, are shown under the columns of BKM \((L+N)\), where \(L\) and \(N\) are respectively the total numbers of boundary and inner nodes. The quantities under the column Exact are the exact solutions. In Figure 1, the small blank circles denote the boundary discretization knots, while the tiny crosses represent the inner knots.

3.1. A 2D homogeneous Helmholtz problem

We first consider the following homogeneous Helmholtz problem

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda^2 u = 0, \quad x, y \in \Omega
\] (19)

subject to the following boundary conditions
\[ u(x, y) = D(x, y), \quad x, y \in S_u, \quad (20) \]

\[ \frac{\partial u(x, y)}{\partial n} = N(x, y), \quad x, y \in S_T. \quad (21) \]

Three cases with analytical solutions given as

\[ u(x, y) = \sin(x)\sin(y), \quad (22a) \]

\[ u(x, y) = x\sin(\sqrt{2}y), \quad (22b) \]

\[ u(x, y) = \sin(\lambda x) + \sin(\lambda y), \quad (22c) \]

are tested. The above Dirichlet and Neumann boundary conditions \( D(x,y) \) and \( N(x,y) \) can be evaluated easily by using the corresponding analytical solutions (22a, b, c). The non-singular solution of the 2D homogeneous Helmholtz operator is

\[ \sigma(r) = J_0(\eta r), \quad (23) \]

where \( J_0 \) is the zero order Bessel function of the first kind and \( \eta \) is the wave number. For the analytical solutions (22a, b, c), \( \eta \) is taken to be \( \sqrt{2}, \sqrt{2} \), and \( \lambda \) respectively. In this study, we take \( \lambda = 10 \).

From the numerical relative errors of the BKM solutions given in Tables 1-1, 1-2, and 1-3, it can be observed that the BKM gives a stable, more accurate, and faster convergent solutions by increasing more internal and external nodes. It is noted that only boundary nodes were required in the homogeneous case. The numerical results indicated that the BKM enjoys a super-convergent property. It is also observed that the BKM results with only 29 nodes were very accurate for the analytical solutions (22a, b) with small wave number, while the analytical solution (22c) needs higher wave number and much more nodes to attain the same accurate BKM results. Taking the consideration of the complicated exterior and interior contours, the accuracy of the proposed BKM solution is very satisfactory.
3.2. An inhomogeneous Helmholtz problem

To investigate the effect of the inner knots to the solution of the inhomogeneous problems, we consider the following problem

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = x, \quad (x, y) \in \Omega, \tag{24}
\]

\[
u(x, y) = D(x, y), \quad (x, y) \in S_u. \tag{25}
\]

\[
\frac{\partial u(x, y)}{\partial n} = N(x, y), \quad (x, y) \in S_T. \tag{26}
\]

The analytical solution is

\[
u(x, y) = \sin x + \sin y + x. \tag{27}
\]

Similar to the previous case, the boundary conditions \(D(x,y)\) and \(N(x,y)\) can be determined from the exact solution (27). Here the rather simple inhomogeneous term is deliberately chosen to show that even for such a smooth linear inhomogeneous term, some inner nodes are also required in finding the BKM solution. The DRM and RBF were employed to evaluate the particular solution. In terms of the multiquadratic (MQ) RBF, the chosen approximated particular solution for equation (9) is

\[
\psi(r) = \left(r^2 + c^2\right)^\nu^2, \tag{28}
\]

where \(c\) is called shape parameter. In solving equation (10), the corresponding MQ-like radial basis function is

\[
\phi(r) = 6\left(r^2 + c^2\right) + \frac{3r^2}{\sqrt{r^2 + c^2}} + \left(r^2 + c^2\right)^\nu^2. \tag{29}
\]
The particular solution is then evaluated by using formulas (8) and (9). Compared with other radial basis functions such as thin plate spline, the MQ enjoys a spectral convergence [22]. However, the accuracy of the solution greatly depends on the optimal value of the shape parameter $c$, which is often problem-dependent [16]. An analytical formula for the value of the optimal $c$ remains an open issue. See Chen and Tanaka [10,15] for some latest development in how to construct an efficient RBFs.

Table 2-1 lists the BKM results without using inner knots. The nearly optimal shape parameters $c$ are obtained by trial-and-error to be 3, 2, 1 for the cases with 21, 25, and 27 knots respectively. It can be observed from the tables that as the node density increases, the optimal value of the shape parameter decreases. It is also observed that these BKM solutions converge unstably and slowly. Unlike the previous homogeneous case, the BKM results were not much improved even with more knots. Table 2-2 gives the solutions by using an additional 7 interior nodes. The locations of these interior knots are listed in the first and second columns of Table 2-2. The BKM solutions at these knots are used to compare their relative errors against the exact ones. In sharp contrast to those without inner knots as given in Table 2-1, it can be observed that the BKM solution by using the additional 7 inner knots has a much faster and stable convergence rate for the present inhomogeneous problem. In addition, the MQ shape parameter $c$ is all set to 9 in these computations but the BKM solution accuracy seems not sensitive to the parameter $c$ if inner nodes are used. Furthermore, the BKM results with only one interior knot is given in Table 2-3. Similar to the case of using 7 inner nodes, the shape parameter of the MQ is all taken as 15 irrespective of the total number of nodes. Comparing the results given in Tables 2-1 to 2-3, we can conclude that the more inner nodes are used, the more accurate and stable the BKM solutions are. Moreover, with the use of inner nodes, the MQ shape parameter in the BKM solution is not sensitive to the node density.

In this inhomogeneous case, it is particularly worth noting that the BKM with one inner knot performed much better than the cases without any inner knot. This indicates that inner nodes are indispensable to guarantee the stability and accuracy in the DRM and RBF evaluation of the particular solutions in the proposed two-steps BKM scheme. Since the DRBEM also applies the DRM to calculate the particular
solution, the argument given in [14] that except for improving the solution accuracy the interior nodes are not necessary in the DRBEM is in doubt.

3.3. A 2D homogeneous convection-diffusion problem

The numerical solution of convection-diffusion problem is often a difficult task due to the troublesome convection terms. It has been claimed that the BEM performs better than the FEM and FDM in solving the convection-diffusion problems due to the fact that the convection terms have been inherently included into the fundamental solution for the convection-diffusion operator. This is also expected to be true to the proposed BKM scheme. For illustration, we consider the applicability of the BKM scheme in using the non-singular general solution of the convection-diffusion operator to the following homogeneous problem

\[ \nabla^2 u = -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \]  

(30)

with boundary conditions

\[ u(x, y) = D(x, y), \quad x, y \in S_u, \]  

(31)

\[ \frac{\partial u(x, y)}{\partial n} = N(x, y), \quad x, y \in S_T. \]  

(32)

The analytical solution is given to be

\[ u(x, y) = e^{-x} + e^{-y}. \]  

(33)

Again the boundary values can be determined from the analytical solution (33). The BKM using the non-singular general solution of Helmholtz operator was also successfully employed to solve the above problem in a smooth elliptical domain and under a Dirichlet boundary condition, where the DRM was used to evaluate the particular solution due to the convection terms [14]. In this study we used the non-singular general solution of the convection-diffusion operator as shown in equation
In addition, the present experiments can handle much more complex-shaped boundary with both Neumann and Dirichlet conditions.

The relative errors of the BKM solutions are summarized in Table 3. It is noted that the BKM results using 17 boundary nodes achieved in average an accuracy up to the fourth significant digits. In contrast, the DRBEM with 16 boundary nodes and 17 interior points [14] produced a less accurate solution for the same homogeneous convection-diffusion problem with a much simpler smooth elliptical domain and Dirichlet boundary conditions. This is believed that this was due to the use of the Laplacian fundamental solution and the BEM with lower order of convergence ratio. It should be pointed out that the present BKM solutions are also better than the BKM solutions with the non-singular solution of Helmholtz operator. This striking accuracy is because the present proposed BKM with the convection-diffusion non-singular solution can well capture the convective effects of the convection-diffusion system.

### 3.4. A 3D homogeneous Helmholtz problem

Three-dimensional problems are usually not easy to deal with partly due to the expensive effort in the mesh generation for mesh-dependent techniques and, more importantly, due to the exponential increasing size of resulting analogous equations. This fact is the so-called curse of dimensionality. The objective of the following experiment is to numerically verify the accuracy and efficiency of the BKM in handling 3D case. Consider

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \lambda^2 u = 0, \quad (x, y, z) \in \Omega, \tag{34}
\]

with the Dirichlet boundary conditions

\[
u(x, y, z) = D(x, y, z), \quad (x, y, z) \in S_u. \tag{35}\]

The two cases with analytical solution are given respectively by
\[ u(x, y, z) = \sin(x) \cos(y) \cos(z), \quad (36a) \]

for an unit sphere domain, and

\[ u(x, y, z) = \sin(\lambda x) + \sin(\lambda y) + \sin(\lambda z), \quad (36b) \]

for a cube domain. The non-singular solutions of 3D homogeneous Helmholtz operator are

\[ \sigma(r) = \frac{\sin(\eta r)}{r}, \quad (37) \]

where \( \eta \) are chosen to be \( \sqrt{3} \) and \( \lambda \) is taken to be 1 in the respective cases. The relative errors of the BKM solutions are tabulated in Tables 4-1 and 4-2. It can be observed from the tables that the BKM worked equally well for this 3D problem and the previous 2D cases. Based on some numerical experiments and theoretical analysis concerning the dimensional effect on the error bounds of the RBF interpolation, Chen and He [23] conjectured that the RBF-based numerical scheme may circumvent the curse of dimensionality like the Monte Carlo method. To be more precise, the computational effort in using the RBF on solving higher-dimensional problems only grows linearly instead of exponentially. Kansa and Hon [17] in their numerical tests also observed that the RBF collocation method seems to enjoy this computational advantage. For numerical verifications, it can be seen from Tables 4-1 and 4-2 that the BKM with only tens points produced rather accurate solutions for the 3D problems. This indicates that the BKM provides a light to tackle high-dimensional problems by using relatively much smaller number of nodes.

3.5. A 3D homogeneous convection-diffusion problem

We expand the previous 2D convection-diffusion problem to a 3D case:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z}, \quad (38) \]
subject to the Dirichlet boundary condition

\[ u(x, y, z) = D(x, y, z), \quad (x, y, z) \in S_u. \]  

(39)

The analytical solution is

\[ u(x, y, z) = e^{-x} + e^{-y} + e^{-z}, \]  

(40)

and the corresponding non-singular general solution is given by

\[ \sigma_n(r) = e^{-\frac{\nu r}{2D}} \frac{\sinh\left(\frac{r\sqrt{3}}{2}\right)}{r}, \]  

(41)

where \( r \) is the distance vector between the source and response nodes and \( \nu \) is the velocity vector \( \{1, 1, 1\}^T \).

Table 5 lists the BKM solution errors against the analytical solutions. The BKM average relative errors at some specified 7 inner nodes with a total of 14 and 22 boundary knots are respectively 1.2e-3 and 4.1e-4. These accurate results numerically illustrate the superior convergence speed and accuracy of the BKM. For comparison, the DRBEM was also applied to solve the same problem with the use of the Laplace fundamental solution and the DRM for the evaluation of the particular solution of convection terms [14]. Although a total of 74 boundary and 27 interior nodes were used in these DRBEM and DRM methods, the accuracy of the solution was inferior to that of the BKM with only 14 nodes due to the fact that the Laplacian does not take the convective terms into the boundary formulation and the BEM suffers from its lower order accuracy. An interesting fact observed from Table 5 is that the more apart of the inner nodes from the boundary, the more accurate is the BKM solution.

4. COMPLETENESS, CONVERGENCE, AND CONDITIONING NUMBER

4.1. Completeness issue
In the case of the $n$-dimensional Helmholtz problem, the complete fundamental solution contains a complex argument [24]:

$$u^*(r) = \frac{1}{4} \left( \frac{\lambda}{2\pi} \right)^{\eta/2-1} \left[ Y_{\eta/2-1}(\lambda r) - iJ_{\eta/2-1}(\lambda r) \right],$$

(42)

where $J$ and $Y$ are respectively the Bessel function of the first and second kinds. The former Bessel function is $C^\infty$ smooth while the latter encounters a singularity at the origin. The non-singular general solution in the BKM can be interpreted as the non-singular imaginary part of the above complete complex singular fundamental solution. This raised the completeness issue of the BKM solution. Chen and Tanaka [11,15] discussed this issue by comparing the BKM to the multiple reciprocity BEM (MRM) where the Laplace fundamental solution is used with high-order terms in which only the singular real part of the complex fundamental solution is applied [25]. For simplicity, DeMey [26] also successfully employed the BEM with the singular real part to calculate the Helmholtz eigenvalue problems that circumvented any complex calculation. However, as was pointed out in [27, 28], the BEM with only the real part of the Helmholtz complex fundamental solution may converge to spurious eigenvalues in some cases. Chen et al [28] also provided some remedies to cure this inefficiency. In fact, Kamiya and Andoh [25] pointed out that the MRM with the Laplacian fundamental solution does not satisfy the Sommerfeld radiation conditions at infinity. Power [29] gave the incompleteness issue of the MRM in performing the Brinkman equation.

The incompleteness concerns on the MRM and BEM with the real part of the Helmholtz fundamental solution also apply to the BKM solution of the Helmholtz problem. However, we note that the eigensolutions of a practical Helmholtz problem can usually be expressed by smooth Bessel function of the first kind rather than singular Bessel function of the second kind. For example, the exact Helmholtz eigenfunctions on the unit disk are series of the Bessel functions of the first kind [30, pp. 378-379]. In other words, only the non-singular general solution is used to avoid the singularity at the origin in many practical computations involving symmetrical
circular and cylindrical domain problems. This provides some intuitive explanations on the feasibility of the BKM. The BKM is expected to be applicable to a broader range of Helmholtz problems than the MRM and BEM with the singular real part of the complex fundamental solution. Although a thorough theoretical analysis on this issue is not an easy task, more numerical experiments will be beneficial to the early development stage of the BKM method.

On the other hand, in the case of the modified Helmholtz problems, we note that there is no such completeness issue as in the Helmholtz problem. The non-singular general solution [15] and singular fundamental solution [24] are respectively

\[
\sigma_n(r) = \frac{1}{2\pi} \left( \frac{\lambda}{2\pi \nu} \right)^{(n^2-1)/2} I_{(n/2)-1}(\lambda r), \quad n \geq 2. \quad (43)
\]

and

\[
u_n^*(r) = \frac{1}{2\pi} \left( \frac{\lambda}{2\pi \nu} \right)^{n/2} K_{(n/2)-1}(\lambda r), \quad n \geq 2. \quad (44)
\]

We could not even be sure the completeness of the solutions (42) or (43). The same difficulty occurs in the convection-diffusion case as shown in the previous formulas (11) and (12). An immediate question in both cases is whether or not the singularity is essential to attain the reliable solutions by the boundary-type discretization schemes.

4.2. Convergence and conditioning number

Like all global numerical schemes, the large dense interpolation matrix resulted from using the RBF-based scheme usually suffers from severe ill-conditioning inefficiency [3, 17]. To further investigate this issue, Table 6 displays the average relative errors and conditioning numbers of the BKM solutions for the previous 2D homogeneous Helmholtz and convection-diffusion problems, with the columns under Err and Cond with lower case \( H \) and \( C \) respectively; and \( L \) is the total number of boundary nodes. The average relative error is defined to be
\[ err = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{u_i - \bar{u}_i}{u_i} \right|, \quad (45) \]

where \( N \) is the total number of inner nodes, \( u_i \) and \( \bar{u}_i \) are respectively the exact and BKM solutions at these nodes. The locations of these inner nodes are the same as those indicated in Tables 1-3. All computations were computed on a Dell-PC Pentium III computer using the Microsoft Fortran Power Station 4.0 with double precision. The LU decomposition algorithm was employed to solve the resulting discretised system of equations.

It can be observed from Table 6 that the convergence of the BKM solution was very fast and stable. It is also interesting to note that although the conditioning numbers of the convection-diffusion problem were much larger than those of the Helmholtz problem, the former in general has higher accuracy than the latter. In the case of the MFS, Golberg and Chen [3] obtained a similar computation. They pointed out that the irrelevances between the ill-conditioning of interpolation matrix and the high accuracy of the solution may be due to some inherent cancellation of round-off errors. In addition, we found that in both computational examples, the conditioning numbers did not change rapidly with the increase of the boundary nodes. In particular, unlike other global schemes, the conditioning number of the Helmholtz problem unexpectedly remained stably mild scale despite the increase of the nodes. It is stressed here that we did not apply any special treatment to solve the resulting BKM discretization equations.

5. CONCLUSIONS

This work validated that the BKM consistently produces very accurate solutions for the 2D Helmholtz and convection-diffusion problems with rather complex-shaped interior and exterior contours, which shows that the BKM is insensitive to geometric irregularity. The applicability of the BKM to solve the 3D Helmholtz and convection-diffusion problems is also demonstrated. The BKM solutions are found to uniformly converge to the exact solutions in all testing cases. The illustrative experiments also
manifested that for inhomogeneous problems, some inner nodes were required to guarantee the stable convergence and high accuracy in the DRM and RBF evaluation of the particular solution.

The BKM is a new RBF-based boundary-type discretization technique with the remarkable merits of efficiency, highly accurate, stable, fast convergent, and mathematical simple. Similar to the domain-type RBF-based methods [31], the essential meshless merit does give the BKM an edge over the BEM to easily handle higher-dimensional complicated-geometry problems by using only scattered knots. Moreover, the numerical experiments showed that the BKM can avoid the curse of dimensionality in the solution of the 3D problem. This attractive advantage is also based on the theoretical analysis and experimental findings that the use of higher order radial basis functions can offset the dimensional affect [15, 17, 23].

The advantage of the BKM over the MFS is that the former eliminates the controversial requirement on the artificial boundary which is required in the latter method. The arbitrariness in the choice of the ambiguous fictitious boundary in the MFS may lead to some troublesome concerns in the engineering computations. The BKM is therefore comparatively more promising in the real world computing. Nonetheless, the BKM is still at its early development stage. Some concerns on the completeness, singularity and conditioning of the BKM interpolations are discussed with some open issues raised in this paper. Much more work will be needed to explore the full potential of the method. The applications of the BKM to solve time-dependent and nonlinear problems are presently under investigations.

REFERENCES


