International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

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INT. J. CONTROL, 2003, VOL. 76, NO. 2, 105–115

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Robust stabilization for a class of discrete-time non-linear systems via output feedback: the unified LMI approach

DANIEL W. C. HO†* and GUOPING LU‡

This paper discusses a robust stabilization problem for a class of multi-input and multi-output (MIMO) discrete-time non-linear systems with both state and control inputs containing non-linear perturbations. The problem is solved via static output feedback and dynamic output feedback, respectively. A unified approach is used to cast the problem into a convex optimization involving linear matrix inequalities (LMI), all the controllers can robustly stabilize the systems and maximize the bound on the non-linear perturbations. This paper also extends the output feedback centralized design approach to a class of discrete-time MIMO non-linear decentralized systems, both robust static and dynamic output feedback controllers are obtained.

1. Introduction

Much attention has been paid to the quadratic stabilization theory of discrete-time systems (see Petersen and Hollot 1986, Yaz and Niu 1989, Gu 1994, Halicka and Rosinova 1994, Zeng 1995, Yuan *et al*. 1996, Stipanovic and Siljak 2001). The problem of robust quadratic stabilization is to find a feedback controller such that the closed-loop systems are stable for all admissible parameter perturbations, and the associated Lyapunov function is quadratic and deterministic. The approach has been shown to be effective for dealing with parameter uncertainty by Yuan *et al*. (1996), Stipanovic and Siljak (2001). Based on strictly quasi-convex optimization, robust quadratic stabilizing controllers via linear static output feedback and state feedback are constructed for discrete-time linear systems with uncertainty by Gu (1994). Under some additional constraints on the variable of matrix inequality (see Remarks 5 and 11), the robust stabilization for a class of discrete-time non-linear systems is formulated into a convex optimization via linear matrix inequalities (LMI) developed by Stipanovic and Siljak (2001). In their work, static state feedback law is designed to stabilize the plant and maximize the bound on the non-linear perturbation terms. The approach by Stipanovic and Siljak (2001) is also extended to a class of interconnected systems, a stabilizing feedback law is presented such that the closed-loop systems are maximally robust with respect to the size of the uncertain interconnected terms. In their design, it requires a special structure of matrix variable L (or L_i) in order to 'recover' control gain K (or K_i) from matrix inequality (also see Remarks 5 and 11).

In this paper, a class of multi-input and multi-output (MIMO) discrete-time systems are discussed, which contains non-linear perturbations on both state and control inputs. The systems in this paper are more general than those systems discussed by Stipanovic and Siljak (2001) where no control input perturbation is considered. The objective of this paper is to present a unified approach to design output feedback control design for the discrete-time systems by means of convex optimization procedure, the output feedback control law can robustly stabilize the overall systems and simultaneously maximize the bound on the non-linear perturbations. The contributions of this paper are shown as follows:

(1) The static output feedback law is presented in this paper. As a special case, static state feedback law can be obtained. However, no special structure of matrix variable is required to obtain control gain K. Those controller designs in Stipanovic and Siljak (2001) and Garcia *et al*. (1994) are special cases of this work (see Remarks 4 and 5).

(2) Based on the Luenberger observer design, a dynamic output feedback law is presented in this paper, and it can also be regarded as an extention of the design by Stipanovic and Siljak (2001).

(3) We generalize our method to MIMO interconnected systems, both static output and dynamic output decentralized controllers are designed. The interconnections among subsystems consist of two parts: one is the linear part and the other is the non-linear perturbations for both state and control inputs. The proposed model is more general than the decentralized model discussed by Stipanovic and Siljak (2001).

Received 18 September 2001. Revised 20 August 2002. Accepted 10 September 2002.

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International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2003 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/0020717031000067367

(4) A unified structure of LMI is introduced to construct robust stabilization controllers via various feedback designs for a wide class of discrete-time systems with either matched uncertainty or unmatched uncertainty, respectively (see Remarks 2 and 4).

2. Preliminaries

Consider the following unforced discrete-time non-linear system

$$z(k+1) = A_0 z(k) + g[k,\ z(k)] \tag{1}$$

where $z(k)$ is the system state and A_0 is a constant matrix with appropriate dimensions; $g = g(k, z)$ is a known vector-valued non-linear function and satisfies the following quadratic inequality for all $(k,\ z)$.

$$g'(k, z)g(k,\ z) \le \alpha^2 z' G' G z \tag{2}$$

where G is a constant matrix with appropriate dimension and α is a non-negative constant. Motivated by Stipanovic and Siljak (2001) we introduce the following definition.

Definition 1: System (1) is robustly stable with degree α if the equilibrium $z = 0$ is globally asymptotically stable for all $g(k, z(k))$ satisfying constraint (2).

The following lemma will be used for the proof of Lemma 2.

Lemma 1—S-procedure lemma (Yakubovich 1977): *Let $\Omega_0(x)$ and $\Omega_1(x)$ be two arbitrary quadratic forms over $\mathbf{R}^n$, then $\Omega_0(x) < 0$ for all $x \in \mathbf{R}^n - \{0\}$ satisfying $\Omega_1(x) \le 0$ if and only if there exist $\tau \ge 0$ such that*

$$\Omega_0(x) - \tau \Omega_1(x) < 0, \qquad \forall x \in \mathbf{R}^n - \{0\}$$

For convenience and compactness, let

$$\mathcal{L}(\gamma, Q, \Gamma_1, \Gamma_2) := \begin{pmatrix} -Q & \Gamma_1' & \Gamma_2' \\ \Gamma_1 & I - Q & 0 \\ \Gamma_2 & 0 & -\gamma I \end{pmatrix} \tag{3}$$

where γ is a scalar, Q, Γ and Γ_1 are matrices with appropriate dimensions and I is an identity matrix with an appropriate dimension. Matrix inequality $\mathcal{L}(\gamma, Q, \Gamma_1, \Gamma_2) < 0$ is a unified LMI structure and the notations in (3) will be used throughout this paper. The dimension of (3) will be different according to the control designs of state feedback, static output feedback and dynamic output feedback, respectively.

The following result is fundamental and will be used throughout this paper, which presents a sufficient condition for robust stability of system (1) in the sense of Definition 1. For convenience of discussion, let $\gamma = \alpha^{-2}$.

Lemma 2: *Unforced system* (1) *is robustly stable with degree α if there exists positive definite matrix Q such that the following convex optimization problem is solvable*

$$\left.\begin{array}{rl} \text{minimize} & \gamma \\ \text{subject to} & \mathcal{L}(\gamma, Q, A_0 Q, GQ) < 0 \end{array}\right\} \tag{4}$$

Proof: If (4) has solution Q, let $P = Q^{-1}$, then it follows from the Schur complement lemma (Boyd *et al.* 1994) that (4) implies that $I - P > 0$ and

$$-P + \alpha^2 G'G + A_0'(P^{-1} - I)^{-1} A_0 < 0 \tag{5}$$

Noting

$$(P^{-1} - I)^{-1} = P + P(I - P)^{-1} P \tag{6}$$

then (5) is equivalent to

$$-P + \alpha^2 G'G + A_0' P A_0 + A_0' P (I - P)^{-1} P A_0 < 0 \tag{7}$$

Choose the Lyapunov function candidate as

$$V_k = z'(k) P z(k) \tag{8}$$

then it follows from (7) that for any $z(k) \ne 0$

$$\begin{aligned} & V_{k+1} - V_k - \left[g'g - \alpha^2 z'(k) G' G z(k)\right] \\ &= [A_0 z(k) + g]' P [A_0 z(k) + g] - z'(k) P z(k) \\ &- [g'g - \alpha^2 z'(k) G' G z(k)] \\ &= z'(k)[-P + \alpha^2 G'G + A_0' P A_0 + A_0' P (I - P)^{-1} P A_0] z(k) \\ &- [(I - P)^{-1/2} P A_0 z(k) - (I - P)^{1/2} g]' [(I - P)^{-1/2} P A_0 z(k) \\ &- (I - P)^{1/2} g] \end{aligned} \tag{9}$$

From Lemma 1, we have that for any $z(k) \ne 0$, equation (9) implies that

$$V_{k+1} - V_k < 0 \tag{10}$$

which completes the proof. □

Remark 1: For completeness of the above proof in (9) on parameter scaling, we can also show that

$$V_{k+1} - V_k - \tau\left[g'g - \alpha^2 z'(k) G' G z(k)\right] < 0$$

by replacing $V_k = \tau z'(k) P z(k)$ in (8), where $\tau > 0$

Remark 2: Lemma 2 will be used in the sequel. Compared with Theorem 1 by Stipanovic and Siljak (2001), Lemma 2 is more compact with a reduced dimension of LMI, which is more effective for computation.

3. Static output feedback

Consider a class of MIMO discrete-time non-linear system as follows

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + f[k, x(k), u(k)] \\ y(k) &= Cx(k) \end{aligned}\right\} \tag{11}$$

where $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^m$ and $y(k) \in \mathbf{R}^p$ are the system state, control input and output, respectively; $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are constant matrices; without loss of generality, suppose that $\mathrm{rank}(C) = p$; $f = f(k, x, u)$ is known vector-valued non-linear function and satisfies the following quadratic inequality for all $(k, x, u) \in \mathbf{Z}_0 \times \mathbf{R}^n \times \mathbf{R}^m$, where $\mathbf{Z}_0 = \{0, 1, 2, \ldots\}$

$$f'(k, x, u) f(k, x, u) \le \alpha^2 \left(x'F'Fx + u'H'Hu\right) \tag{12}$$

where F, H are constant matrices with appropriate dimensions and α is a non-negative constant.

Remark 3: We can replace (12) with the constraint

$$f'(k, x, u) f(k, x, u) \le \alpha^2 \left(x'F'Fx + 2x'F'Hu + u'H'Hu\right) \tag{13}$$

It is easy to show from linear algebra that (13) is equivalent to (12). For simplicity, we only make assumption for $f(k, x, u)$ in the form of (12).

Remark 4: If $f[k, x(k), u(k)] = \Delta A(k)x(k) + \Delta B(k)u(k) + f_0[k, x(k), u(k)]$, where the norm of $\Delta A(k)$ and $\Delta B(k)$ are uniformly bounded by two constant scalars or satisfy with matched (or unmatched) conditions, and $f_0(k, x, u)$ is global Lipschitz on x and u with $f_0(k, 0, 0) = 0$ for any $k \in \mathbf{Z}_0$. Then this class of Lipschitz system with uncertainty can be included in system (11). The approach in this paper may include analysis of a wider class of systems but S-procedure is not straightforward. In addition, system (11) is more general than the following systems: (i) the discrete-time linear system with structure uncertainties discussed by Garcia *et al*. (1994); (ii) the perturbed discrete-time system with no control input perturbation considered by Stipanovic and Siljak (2001).

In this section, we consider the following form of linear static output feedback controller

$$u(k) = Ky(k) \tag{14}$$

where $K \in \mathbf{R}^{m\times p}$ is a constant matrix to be determined.

If there exists a controller in the form of (14) such that the closed-loop systems (11) and (14) are robustly stable with degree α in the sense of Definition 1, then we say that system (11) can be robustly stabilizable with degree α by means of controller (14).

The following theorem presents a way to construct static output feedback law (14) in which sufficient condition is presented by means of LMI.

Theorem 1: *System* (11) *is robustly stabilizable with degree* α *by means of static output feedback* (14) *if the following optimization problem on matrices* $Q \in \mathbf{R}^{n\times n}$, $X \in \mathbf{R}^{m\times p}$ *and* $Z \in \mathbf{R}^{p\times p}$ *is solvable*

$$\begin{aligned} &\textit{minimize} \quad \gamma \\ &\textit{subject to} \quad \mathcal{L}\left(\gamma, Q, AQ + BXC, \begin{pmatrix} FQ \\ HXC \end{pmatrix}\right) < 0 \\ &\qquad CQ = ZC \end{aligned} \tag{15}$$

Proof: The resulting closed-loop systems of (11) and (14) are

$$x(k+1) = (A + BKC)x(k) + f[k, x(k), KCx(k)] \tag{16}$$

where

$$f'[k, x(k), KCx(k)] f[k, x(k), KCx(k)] \le \alpha^2 x'(k) \times \begin{pmatrix} F \\ HKC \end{pmatrix}' \begin{pmatrix} F \\ HKC \end{pmatrix} x(k) \tag{17}$$

That is, system (16) is equivalent to (1) with constraint (2) if

$$A_0 = A + BKC, \quad G = \begin{pmatrix} F \\ HKC \end{pmatrix} \tag{18}$$

Noting that the full row rank of C and Q is a positive-definite matrix from LMI (15), then matrix equation $CQ = ZC$ implies

$$\begin{aligned} p &\ge \mathrm{rank}(Z) \ge \mathrm{rank}(ZC) = \mathrm{rank}(CQ) \\ &\ge \mathrm{rank}\left[(CQ)Q^{-1}\right] = \mathrm{rank}(C) = p \end{aligned} \tag{19}$$

that is, Z is non-singular. If the gain of control law in the form of (14) can be chosen as

$$K = XZ^{-1} \tag{20}$$

then, in LMI (4

$$\begin{aligned} \Gamma_1 &:= A_0 Q = (A + BKC)Q = AQ + BKCQ \\ &= AQ + BKZC = AQ + BXC, \\ \Gamma_2 &:= GQ = \begin{pmatrix} F \\ HKC \end{pmatrix} Q = \begin{pmatrix} FQ \\ HXC \end{pmatrix} \end{aligned} \tag{21}$$

then for the closed-loop systems (16) with (20), the following LMI is equivalent to the LMI in (4)

$$\mathcal{L}\left(\gamma, Q, AQ + BXC, \begin{pmatrix} FQ \\ HXC \end{pmatrix}\right) < 0$$

From Lemma 2, the closed-loop systems (16) are robustly stable with degree α, which completes the proof. □

As a direct application of Theorem 1, choosing $C = I$ in Theorem 1, then matrix equation $CQ = ZC$ holds automatically if $Z = Q$, that is, the constraint $CQ = ZC$ is equivalent to $Z = Q$ in this case. Therefore we have the following result, which presents a sufficient condition in the form of LMI under which system can be robustly stabilized via static state feedback law.

Corollary 1: *System* (11) *is robustly stabilizable with degree* α *by means of static state feedback if the following convex optimization problem on matrices* $Q \in \mathbf{R}^{n\times n}$ *and* $X \in \mathbf{R}^{m\times n}$ *is solvable*

$$\begin{aligned} &\textit{minimize} \quad \gamma \\ &\textit{subject to} \quad \mathcal{L}\left(\gamma, Q, AQ + BX, \begin{pmatrix} FQ \\ HX \end{pmatrix}\right) < 0 \end{aligned} \tag{22}$$

In this case, a static state feedback law can be given as

$$u(k) = Kx(k) = XQ^{-1}x(k) \tag{23}$$

Remark 5: In order to 'recover' control gain K for state feedback of single-input system, sti sti introduce a special structure matrix variable L to guarantee the resulting matrix inequality to be an LMI and then K is obtained from L (see (19)–(21) in Stipanovic and Siljak, 2001). Therefore Corollary 1 presents a more efficient approach to 'recover' K explicitly in (23) via less conservative LMI (22) (see also Example 1). In addition, Theorem 2 given by Stipanovic and Siljak (2001) is a special case of Theorems 1 and 2. Furthermore, the result in this section can be regarded as an extention of that by Garcia *et al.* (1994) and Gu (1994) where state feedback and static output feedback are obtained by means of Riccati equation approach and quasiconvex optimization approach, respectively.

Remark 6: An assumption on (A, B) controllable is made by Stipanovic and Siljak (2001). However this assumption is not introduced in this paper. It is due to that if LMI (15) or (27) holds, it can guarantee the robust stability of the resulting closed-loop systems. In fact, it can be observed that (A, B) stabilizable can be derived from the associated LMI.

Since the optimization problem (15) contains the constraint of matrix equation $CQ = ZC$, MATLAB LMI Toolbox (Gahinet *et al.* 1995) is hard to solve (15) directly. In order to convert the optimization problem into an LMI, we shall show that this constraint on Q and Z can be transformed into an equivalent constraint on Q, then the optimization problem (15) will be equivalent to an LMI.

For convenience, we present the singular value decomposition of C as

$$C = U(\,C_0 \quad 0\,)V' \tag{24}$$

where $U \in \mathbf{R}^{p\times p}$ and $V \in \mathbf{R}^{n\times n}$ are unitary matrices and $C_0 \in \mathbf{R}^{p\times p}$ is a diagonal matrix with positive diagonal elements in decreasing order.

The following lemma presents an equivalent condition on matrix equation $CQ = ZC$.

Lemma 3: *For a given* $C \in \mathbf{R}^{p\times n}$ *with rank* $(C) = p$, *assume that* $Q \in \mathbf{R}^{n\times n}$ *is a symmetric matrix, then there exists a matrix* $Z \in \mathbf{R}^{p\times p}$ *such that* $CQ = ZC$ *if and only if*

$$Q = V\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}V'$$

where $Q_1 \in \mathbf{R}^{p\times p}$, $Q_2 \in \mathbf{R}^{(n-p)\times(n-p)}$.

Proof: If $p = n$, from the proof of Theorem 1, C is non-singular, it is clear that the result is true for solvable on Z. Without loss of generality, suppose $p < n$. From $CQ = ZC$ and the singular value decomposition of C, that is, $C = U(\,C_0 \ 0\,)V'$, we have that matrix equation $CQ = ZC$ is equivalent to $U(\,C_0 \ 0\,)V'Q = ZU(\,C_0 \ 0\,)V'$. That is

$$(\,UC_0 \quad 0\,)V'QV = (\,ZUC_0 \quad 0\,) \tag{25}$$

Suppose

$$Q = V\begin{pmatrix} Q_1 & Q_0' \\ Q_0 & Q_2 \end{pmatrix}V'$$

where $Q_1 \in \mathbf{R}^{p\times p}$, $Q_2 \in \mathbf{R}^{(n-p)\times(n-p)}$ and $Q_0 \in \mathbf{R}^{(n-p)\times p}$, then (25) is equivalent to

$$(\,UC_0Q_1 \quad UC_0Q_0\,) = (\,ZUC_0 \quad 0\,) \tag{26}$$

Matrix equation (26) is solvable on Z if and only if $UC_0Q_0 = 0$, that is, $Q_0 = 0$, which completes the proof. □

From Theorem 1 and Lemma 3, we have the following result.

Theorem 2: *System* (11) *is robustly stabilizable with degree* α *by static output feedback law if the following convex optimization problem on matrices* $Q_1 \in \mathbf{R}^{p\times p}$, $Q_2 \in \mathbf{R}^{(n-p)\times(n-p)}$, $X \in \mathbf{R}^{m\times p}$ *is solvable*

$$\left.\begin{aligned} &\textit{minimize} \quad \gamma \\ &\textit{subject to} \quad \mathcal{L}\left(\gamma, V\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}V', \Gamma_1, \Gamma_2\right) < 0 \end{aligned}\right\} \tag{27}$$

where

$$\left.\begin{aligned}\Gamma_1 &= AV\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}V' + BXC, \\ \Gamma_2 &= \begin{pmatrix} FV\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}V' \\ HXC \end{pmatrix}\end{aligned}\right. \tag{28}$$

In this case, a static output feedback controller of form (14) *can be chosen as*

$$u(k) = XUC_0Q_1^{-1}C_0^{-1}U'y(k) \tag{29}$$

Remark 7: In controller form (29), U and C_0 are determined by the singular value decomposition of C (see (24)), X and Q_1 are the solutions of LMI (27) and (28).

Remark 8: The control gain (20) involves Z and is not solvable by the existing LMI tools (Gahsinet *et al.* (1995) since the optimization problem (15) is an LMI with an additional matrix equality constraint. The relationship between Z and Q (from $CQ = ZC$) imposes a special structure on Q, which is equivalent to an explicit form of

$$Q = V\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}V'$$

as shown in Lemma 3. As a result, an explicit control law (29), dependent only on LMI variables, is obtained in Theorem 2.

4. Dynamic output feedback

In this section, we consider stabilization for system (11) via the following Luenberger-like dynamic output feedback controller

$$\left.\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k)) \\ u(k) &= K\hat{x}(k)\end{aligned}\right\} \tag{30}$$

where $L \in \mathbf{R}^{n\times p}$, $K \in \mathbf{R}^{m\times n}$ are two constant matrices to be determined.

Let the difference of $x(k)$ and $\hat{x}(k)$ be $e(k)$, that is, $e(k) = x(k) - \hat{x}(k)$, then the closed-loop systems of (11) and (30) are in the form of (1) with

$$\begin{aligned} z(k) &= \begin{pmatrix} \hat{x}(k) \\ e(k) \end{pmatrix}, \quad A_0 = \begin{pmatrix} A + BK & LC \\ 0 & A - LC \end{pmatrix}, \\ g &= \begin{pmatrix} 0 \\ f[k, \hat{x}(k) + e(k), K\hat{x}(k)] \end{pmatrix}\end{aligned} \tag{31}$$

Theorem 3: *System* (11) *is robustly stable with degree* α *via dynamic output feedback in the form of* (30) *if there exist matrices* $Q_1, Q_2 \in \mathbf{R}^{n\times n}$, $X \in \mathbf{R}^{m\times n}$, $Y \in \mathbf{R}^{n\times p}$ *and* $Z \in \mathbf{R}^{p\times p}$ *such that the following optimization problem is solvable*

$$\left.\begin{aligned} &\textit{minimize} && \gamma \\ &\textit{subject to} && \mathcal{L}(\gamma, Q, \Gamma_1, \Gamma_2) < 0 \\ &&& CQ_2 = ZC \end{aligned}\right\} \tag{32}$$

where

$$\begin{aligned} Q &= \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} AQ_1 + BX & YC \\ 0 & AQ_2 - YC \end{pmatrix} \\ \Gamma_2 &= \begin{pmatrix} FQ_1 & FQ_2 \\ HX & 0 \end{pmatrix}\end{aligned} \tag{33}$$

In this case, a dynamic output feedback controller can be given by (30) *with*

$$L = YZ^{-1}, \quad K = XQ_1^{-1} \tag{34}$$

Proof: From (12), we have

$$\begin{aligned} g'g &= f'[k, \hat{x}(k) + e(k), K\hat{x}(k)]f[k, \hat{x}(k) + e(k), K\hat{x}(k)] \\ &\le \alpha^2\{[\hat{x}(k) + e(k)]'F'F[\hat{x}(k) + e(k)] \\ &\quad + \hat{x}'(k)K'H'HK\hat{x}(k)\} \\ &= \alpha^2 z'(k)\begin{pmatrix} F & F \\ HK & 0 \end{pmatrix}'\begin{pmatrix} F & F \\ HK & 0 \end{pmatrix}z(k) \end{aligned} \tag{35}$$

That is, in (2)

$$G = \begin{pmatrix} F & F \\ HK & 0 \end{pmatrix} \tag{36}$$

From (31)–(34) and (36), we have

$$\left.\begin{aligned} A_0Q &= \begin{pmatrix} A + BK & LC \\ 0 & A - LC \end{pmatrix}\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \\ &= \begin{pmatrix} AQ_1 + BKQ_1 & LCQ_2 \\ 0 & AQ_2 - LCQ_2 \end{pmatrix} \\ &= \begin{pmatrix} AQ_1 + BKQ_1 & LZC \\ 0 & AQ_2 - LZC \end{pmatrix} \\ &= \begin{pmatrix} AQ_1 + BX & YC \\ 0 & AQ_2 - YC \end{pmatrix} \\ &= \Gamma_1 \\ GQ &= \begin{pmatrix} F & F \\ HK & 0 \end{pmatrix}\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} = \begin{pmatrix} FQ_1 & FQ_2 \\ HK & 0 \end{pmatrix} \\ &= \Gamma_2 \end{aligned}\right\} \tag{37}$$

Then it follows from Lemma 2 that the resulting closed-loop systems (11) and (30) with (34), that is, system (1) with the parameters given in the form of (31) and (34), are robustly stable, which completes the proof. □

Similar to Theorem 2, we have the following result from Lemma 3.

Theorem 4: *System* (11) *is robustly stable with degree* α *via dynamic output feedback in the form of* (30) *if there exist matrices* $Q_1 \in \mathbf{R}^{n\times n}$, $Q_{21} \in \mathbf{R}^{p\times p}$, $Q_{22} \in \mathbf{R}^{(n-p)\times(n-p)}$, $X \in \mathbf{R}^{m\times n}$ *and* $Y \in \mathbf{R}^{n\times p}$ *such that the following convex optimization problem is solvable.*

$$\left.\begin{aligned} &\textit{minimize} \quad \gamma \\ &\textit{subject to} \quad \mathcal{L}(\gamma, Q, \Gamma_1, \Gamma_2) < 0 \end{aligned}\right\} \tag{38}$$

where

$$\left.\begin{aligned} Q &= \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \quad Q_2 = V\begin{pmatrix} Q_{21} & 0 \\ 0 & Q_{22} \end{pmatrix}V' \\ \Gamma_1 &= \begin{pmatrix} AQ_1 + BX & YC \\ 0 & AQ_2 - YC \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} FQ_1 & FQ_2 \\ HX & 0 \end{pmatrix} \end{aligned}\right\} \tag{39}$$

In this case, a dynamic output feedback controller can be given by (30) *with*

$$L = YUC_0Q_{21}^{-1}C_0^{-1}U', \quad K = XQ_1^{-1} \tag{40}$$

The outline of proof: It follows from (24) and $CQ_2 = ZC$ in (32) that $U(\,C_0 \;\, 0\,)V'Q_2 = ZU(\,C_0 \;\, 0\,)V'$, that is, $(\,UC_0 \;\, 0\,)V'Q_2V = (\,ZUC_0 \;\, 0\,)$. From (39), we have that $UC_0Q_{21} = ZUC_0$, that is,

$$Z = UC_0Q_{21}C_0^{-1}U' \tag{41}$$

Therefore, (40) can be obtained by (32) and (41). □

Remark 9: Theorems 3 and 4 can be regarded as an extention of Theorems 1 and 2, and can also be regarded as an extention of the results by Stipanovic and Slijak (2001).

5. Decentralized control for interconnected discrete-time systems

Consider a class of large-scale discrete-time systems $\mathcal{S}$ composed of N interconnected subsystem $\mathcal{S}_i$. Each subsystem $\mathcal{S}_i$ is described as

$$\left.\begin{aligned} x_i(k+1) &= \sum_{j=1}^{N} A_{ij}x_j(k) + B_iu_i(k) + f_i[k, x(k), u(k)] \\ y_i(k) &= C_ix_i(k), \qquad i = 1, 2, \ldots, N \end{aligned}\right\} \tag{42}$$

where $x_i(k) \in \mathbf{R}^{n_i}$, $u_i(k) \in \mathbf{R}^{m_i}$, and $y_i(k) \in \mathbf{R}^{p_i}$ are the sub-state, sub-control, and sub-output vectors, respectively. A_{ij}, B_i and C_i denote the system matrix, input matrix and output matrix with appropriate dimensions, respectively. $x = (x_1', x_2', \ldots, x_N')' \in \mathbf{R}^n$, $u = (u_1', u_2', \ldots, u_N')' \in \mathbf{R}^m$, $n = \sum_{i=1}^N n_i$, $m = \sum_{i=1}^N m_i$, $p = \sum_{i=1}^N p_i$. Moreover, all non-linear interconnection functions are assumed to satisfy the quadratic constraints for all $(k, x) \in \mathbf{Z}_0 \times \mathbf{R}^n$

$$\begin{aligned} f_i'(k, x, u)f_i(k, x, u) &\le \alpha_i^2\big(x'F_i'F_ix + u'H_i'H_iu\big), \\ i &= 1, 2, \ldots, N, \end{aligned} \tag{43}$$

where F_i and H_i are constant matrices with appropriate dimension, α_i is non-negative constant. The overall interconnected systems can be rewritten in a compact form

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + f[k, x(k), u(k)] \\ y(k) &= Cx(k) \end{aligned}\right\} \tag{44}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the input of the system, $y = (y_1', y_2', \ldots, y_N')' \in \mathbf{R}^p$ is the output and $A = \text{block}\big(A_{ij}\big)_{N\times N}$, $B = \text{diag}\{B_1, B_2, \ldots, B_N\}$ and $C = \text{diag}\{C_1, C_2, \ldots, C_N\}$ are constant matrices with appropriate dimensions. Without loss of generality, C is assumed to be full rank. In (44), the interconnection function $f : \mathbf{Z}_0 \times \mathbf{R}^n \to \mathbf{R}^n$, $f = (f_1', f_2', \ldots, f_N')'$ is constrained for all $(k, x, u) \in \mathbf{Z}_0 \times \mathbf{R}^n$ as

$$f'(k, x, u)f(k, x, u) \le \sum_{i=1}^{N} \alpha_i^2\big(x'F_i'F_ix + u'H_i'H_iu\big) \tag{45}$$

Before presenting the control design in this section, a useful lemma can be obtained by Lemma 2. Consider system (1) with the constraint

$$g'(k, z)g(k, z) \le z'\left(\sum_{i=1}^{N} \alpha_i^2 G_i'G_i\right)z \tag{46}$$

where G_i is constant matrix with appropriate dimension, α_i is non-negative constant, $i = 1, 2, \ldots, N$.

For convenience of discussion, let $\gamma_i = \alpha_i^{-2}$. Similar to Lemma 2, we have the following useful lemma.

Lemma 4: *System* (1) *with constraint* (46) *is robustly stable if there exists positive definite matrix* Q *such that the following convex optimization problem is solvable*

$$\left.\begin{aligned} &\textit{minimize} \quad \gamma_1 + \gamma_2 + \cdots + \gamma_N \\ &\textit{subject to} \quad \mathcal{L}_D(\Gamma_0, Q, \Gamma_1, \Gamma_2) < 0 \end{aligned}\right\} \tag{47}$$

where $\Gamma_1 = A_0 Q$, $\Gamma_2 = GQ = \begin{pmatrix} G'_1 & G'_2 & \ldots & G'_N \end{pmatrix}' Q$, $\Gamma_0 = \operatorname{diag}\{\gamma_1 I_1, \ldots, \gamma_N I_N\}$, $I_1, \ldots, I_N$ are identity matrices with appropriate dimensions, and

$$\mathcal{L}_D(\Gamma_0, Q, \Gamma_1, \Gamma_2) := \begin{pmatrix} -Q & \Gamma'_1 & \Gamma'_2 \\ \Gamma_1 & I - Q & 0 \\ \Gamma_2 & 0 & -\Gamma_0 \end{pmatrix} \tag{48}$$

In this case, we call that system (1) is of *robust stability with degree vector* $(\alpha_1, \ldots, \alpha_N)$.

Remark 10: System (42) is more general than the interconnected model (35) discussed by Stipanovic and Siljak (2001). It is because the system (42) with $C_i = I_i$, $A_{ij} = 0$ for $i \neq j$, $H_i = 0$ and $m_i = 1$ is the same as those discussed by Stipanovic and Siljak (2001), that is, no linear interconnected terms A_{ij} $(i \neq j)$ and no control input perturbation are considered by Stipanovic and Siljak (2001).

5.1. *Static output feedback*

As a counterpart of § 3, in this subsection, we present a decentralized static output feedback controller design as

$$u(k) = Ky(k) = \operatorname{diag}\{K_1, K_2, \ldots, K_N\} \begin{pmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_N(k) \end{pmatrix} \tag{49}$$

where matrices $K_i \in \mathbf{R}^{m_i \times p_i}$ to be determined, $i = 1, 2, \ldots, N$.

For convenience, let $\gamma_i = \alpha_i^{-2}$, $i = 1, 2, \ldots, N$. Similar to Theorem 1, the following theorem presents an explicit way to construct decentralized static output feedback controller law (49), in which sufficient condition is presented by means of LMI.

Theorem 5: *System* (44) *is robustly stabilizable with degree vector* $(\alpha_1, \ldots, \alpha_N)$ *by means of decentralized static output feedback* (49) *if the following optimization problem on matrices* $Q_i \in \mathbf{R}^{n_i \times n_i}$, $X_i \in \mathbf{R}^{m_i \times p_i}$ *and* $Z_i \in \mathbf{R}^{p_i \times p_i}$ $(i = 1, 2, \ldots, N)$ *is solvable*

$$\left.\begin{aligned} &\textit{minimize} \quad \gamma_1 + \gamma_2 + \cdots + \gamma_N \\ &\textit{subject to} \quad \mathcal{L}_D(\Gamma_0, Q, \Gamma_1, \Gamma_2) < 0, \\ &\qquad C_i Q_i = Z_i C_i, \quad i = 1, 2, \ldots, N \end{aligned}\right\} \tag{50}$$

where

$$\left.\begin{aligned} &\Gamma_0 = \operatorname{diag}\{\gamma_1 I_1, \ldots, \gamma_N I_N\}, \quad \Gamma_1 = AQ + BXC \\ &\Gamma_2 = \Big(\begin{pmatrix} QF'_1 & C'X'H'_1 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} QF'_N & C'X'H'_N \end{pmatrix} \Big)' \\ &Q = \operatorname{diag}\{Q_1, Q_2, \ldots, Q_N\}, \; X = \operatorname{diag}\{X_1, X_2, \ldots, X_N\} \\ &Z = \operatorname{diag}\{Z_1, Z_2, \ldots, Z_N\} \end{aligned}\right\} \tag{51}$$

In this case, the static output feedback law can be given as

$$u_i(k) = X_i Z_i^{-1} y_i(k), \quad i = 1, 2, \ldots, N \tag{52}$$

Proof: Similar to the proof of Theorem 1, Z_i is nonsingular for $i = 1, 2, \ldots, N$, which implies that (52) is well-defined. Then the closed-loop systems of (44), (49) and (52) can be rewritten in the form of (1) with constraint (46)

$$\left.\begin{aligned} &A_0 = A + BKC, \quad g[k, x(k)] = f[k, x(k), KCx(k)] \\ &K = \operatorname{diag}\{K_1, \ldots, K_N\} = \operatorname{diag}\{X_1 Z_1^{-1}, \ldots, X_N Z_N^{-1}\} \end{aligned}\right\} \tag{53}$$

In addition, from (43) we have

$$g'[k, x(k)] g[k, x(k)] \leq x'(k) \sum_{i=1}^{N} \alpha_i^2 \begin{pmatrix} F_i \\ H_i KC \end{pmatrix}' \begin{pmatrix} F_i \\ H_i KC \end{pmatrix} x(k) \tag{54}$$

That is

$$G_i = \begin{pmatrix} F_i \\ H_i KC \end{pmatrix}, \quad i = 1, 2, \ldots, N \tag{55}$$

Noticing that $C_i Q_i = Z_i C_i$, $K_i = X_i Z_i^{-1}$ for $i = 1, 2, \ldots, N$, then $CQ = ZC$ and $KZ = X$, which implies that $KCQ = KZC = XC$. Therefore

$$\left.\begin{aligned} &A_0 Q = (A + BKC)Q = AQ + BKCQ = AQ + BXC = \Gamma_1 \\ &\begin{pmatrix} G_1 \\ \vdots \\ G_N \end{pmatrix} Q = \begin{pmatrix} \begin{pmatrix} F_1 Q \\ H_1 KCQ \end{pmatrix} \\ \vdots \\ \begin{pmatrix} F_N Q \\ H_N KCQ \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} F_1 Q \\ H_1 XC \end{pmatrix} \\ \vdots \\ \begin{pmatrix} F_N Q \\ H_N XC \end{pmatrix} \end{pmatrix} = \Gamma_2 \end{aligned}\right\} \tag{56}$$

Then the LMI in (50) is equivalent to the LMI in (47) and it follows from Lemma 4 that the resulting closed-loop systems are robustly stable, which completes the proof. □

Choose $C_i = I_i$, that is $y_i = x_i$ for $i = 1, 2, \ldots, N$, then the following result can be directly obtained from Theorem 5.

Corollary 1: *System* (44) *is robustly stabilizable with degree vector* $(\alpha_1, \ldots, \alpha_N)$ *by means of decentralized static state feedback if the following convex optimization problem on matrices* $Q_i \in \mathbf{R}^{n_i \times n_i}$ *and* $X_i \in \mathbf{R}^{m_i \times n_i}$ $(i = 1, 2, \ldots, N)$ *is solvable*

$$\left.\begin{aligned} &\textit{minimize} \quad \gamma_1 + \gamma_2 + \cdots + \gamma_N \\ &\textit{subject to} \quad \mathcal{L}_D(\Gamma_0, Q, \Gamma_1, \Gamma_2) < 0 \end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned} &\Gamma_0 = \mathrm{diag}\{\gamma_1 I_1, \ldots, \gamma_N I_N\}, \quad \Gamma_1 = AQ + BX \\ &\Gamma_2 = \big(\begin{pmatrix} QF_1' & X'H_1' \end{pmatrix} \quad \cdots \quad \begin{pmatrix} QF_N' & X'H_N' \end{pmatrix} \big)' \\ &Q = \mathrm{diag}\{Q_1, Q_2, \ldots, Q_N\}, \ X = \mathrm{diag}\{X_1, X_2, \ldots, X_N\} \end{aligned}\right\} \tag{58}$$

In this case, the static state feedback law can be given as follows

$$u_i(k) = X_i Q_i^{-1} x_i(k), \quad i = 1, 2, \ldots, N \tag{59}$$

Remark 11: Similar to Remark 5, the decentralized control law in Corollary 2 can be obtained directly from the solution of a convex optimization problem, which is less conservative than the results given by Stipanovic and Siljak (2001). In their work, some additional constraints (see L_i and $\tilde{P}_i$ in (39)–(41) by Stipanovic and Siljak (2001)) are required to 'recover' control gain K_i from the relating matrix inequality even in this special case.

Suppose that the singular value decomposition of C_i is

$$C_i = U_i \begin{pmatrix} C_{0i} & 0 \end{pmatrix} V_i' \tag{60}$$

where $U_i \in \mathbf{R}^{p_i \times p_i}$ and $V_i \in \mathbf{R}^{n_i \times n_i}$ are unitary matrices and $C_{0i} \in \mathbf{R}^{p_i \times p_i}$ is a diagonal matrix with positive diagonal elements in decreasing order, $i = 1, 2, \ldots, N$.

Similar to Theorem 2, we have the following result from Lemma 3.

Theorem 6: *System* (44) *is robustly stabilizable with degree vector* $(\alpha_1, \ldots, \alpha_N)$ *by means of decentralized static output feedback* (49) *if the following convex optimization problem on matrices* $Q_{1i} \in \mathbf{R}^{p_i \times p_i}$, $Q_{2i} \in \mathbf{R}^{(n_i - p_i) \times (n_i - p_i)}$, $X_i \in \mathbf{R}^{m_i \times p_i}$ *is solvable*

$$\left.\begin{aligned} &\textit{minimize} \quad \gamma_1 + \gamma_2 + \cdots + \gamma_N \\ &\textit{subject to} \quad \mathcal{L}_D(\Gamma_0, Q, \Gamma_1, \Gamma_2) < 0 \end{aligned}\right\} \tag{61}$$

where

$$\left.\begin{aligned} &\Gamma_0 = \mathrm{diag}\{\gamma_1 I_1, \ldots, \gamma_N I_N\}, \qquad \Gamma_1 = AQ + BXC \\ &\Gamma_2 = \big(\begin{pmatrix} QF_1' & C'X'H_1' \end{pmatrix} \quad \cdots \quad \begin{pmatrix} QF_N' & C'X'H_N' \end{pmatrix} \big)' \\ &Q = \mathrm{diag}\left\{ V_1 \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{12} \end{pmatrix} V_1', \ldots, V_N \begin{pmatrix} Q_{N1} & 0 \\ 0 & Q_{N2} \end{pmatrix} V_N' \right\} \\ &X = \mathrm{diag}\{X_1, X_2, \ldots, X_N\} \end{aligned}\right\} \tag{62}$$

In this case, the static output feedback law can be given as

$$u_i(k) = X_i U_i C_{0i} Q_{i1}^{-1} C_{0i}^{-1} U_i' y_i(k), \qquad i = 1, 2, \ldots, N \tag{63}$$

5.2. *Dynamic output feedback*

In this subsection, we consider stabilization for system (44) or (42) via the following Luenberger-like decentralized dynamic output feedback controller

$$\left.\begin{aligned} &\hat{x}_i(k+1) = A_{ii}\hat{x}_i(k) + B_i u_i(k) + L_i[y_i(k) - C_i\hat{x}_i(k)] \\ &u_i(k) = K_i \hat{x}_i(k) \end{aligned}\right\} \tag{64}$$

where $L_i \in \mathbf{R}^{n_i \times p_i}$ and $K_i \in \mathbf{R}^{m_i \times n_i}$ $(i = 1, 2, \ldots, N)$ are parameter matrices to be determined.

Let $e_i(k) = x_i(k) - \hat{x}_i(k)$, $z_i(k) = \begin{pmatrix} \hat{x}_i'(k) & e_i'(k) \end{pmatrix}'$, $i = 1, 2, \ldots, N$; and

$$\left.\begin{aligned} \hat{x}(k) &= \begin{pmatrix} x_1'(k) & x_2'(k) & \cdots & x_N'(k) \end{pmatrix}' \\ e(k) &= \begin{pmatrix} e_1'(k) & e_2'(k) & \cdots & e_N'(k) \end{pmatrix}' \\ z(k) &= \begin{pmatrix} z_1'(k) & z_2'(k) & \cdots & z_N'(k) \end{pmatrix}' \end{aligned}\right\} \tag{65}$$

then after some manipulation, the closed-loop systems of (42) and (64) can be rewritten in the form of (1) with constraint (46)

$$\left.\begin{aligned} &A_0 = (\tilde{A}_{ij})_{N\times N}, \qquad g[k, z(k)] = \mathrm{block}(g_i[k, z(k)])_{N\times 1} \\ &\tilde{A}_{ii} = \begin{pmatrix} A_{ii} + B_i K_i & L_i C_i \\ 0 & A_{ii} - L_i C_i \end{pmatrix} \\ &\tilde{A}_{ij} = \begin{pmatrix} 0 & 0 \\ A_{ij} & A_{ij} \end{pmatrix}, \ i \neq j \\ &g_i[k, z(k)] = \begin{pmatrix} 0 \\ f_i[k, \hat{x}(k) + e(k), K\hat{x}(k)] \end{pmatrix} \\ &K = \mathrm{diag}\{K_1, \ldots, K_N\} \end{aligned}\right\} \tag{66}$$

In this case, after some manipulations, $g[k, x(k)]$ is satisfied with constraint (46), where

$$G_i = \begin{pmatrix} F_i \operatorname{diag}\{(I_1 \quad I_1), \ldots, (I_N \quad I_N)\} \\ H_i \operatorname{diag}\{K_1(I_1 \quad 0), \ldots, K_N(I_N \quad 0)\} \end{pmatrix} \tag{67}$$

Similar to Theorems 2, 4 and 6, we have the following results from Lemma 3.

Theorem 7: *System* (44) *is robustly stabilizable with degree vector* $(\alpha_1, \ldots, \alpha_N)$ *by means of decentralized dynamic output feedback* (64) *if the following convex optimization problem on matrices* $Q_{i1} \in \mathbf{R}^{n_i \times n_i}$, $Q_{i21} \in \mathbf{R}^{p_i \times p_i}$, $Q_{i22} \in \mathbf{R}^{(n_i - p_i)\times(n_i - p_i)}$, $X_i \in \mathbf{R}^{m_i \times n_i}$ *and* $Y_i \in \mathbf{R}^{n_i \times p_i}$ *(*$i = 1, 2, \ldots, N$*) is solvable*

$$\left.\begin{aligned} &\textit{minimize} \quad \gamma_1 + \gamma_2 + \cdots + \gamma_N \\ &\textit{subject to} \quad \mathcal{L}_D(\Gamma_0, Q, \Gamma_1, \Gamma_2) < 0 \end{aligned}\right\} \tag{68}$$

where $Q_{i2} = V_i \operatorname{diag}\{Q_{i21}, Q_{i22}\} V_i'$, $Q_i = \operatorname{diag}\{Q_{i1}, Q_{i2}\}$

$$\left.\begin{aligned} &Q = \operatorname{diag}\{Q_1, Q_2, \cdots, Q_N\}, \qquad \Gamma_1 = \operatorname{block}(\Gamma_{1ij})_{N\times N}, \\ &\Gamma_2 = \operatorname{block}(\Gamma_{2i})_{N\times 1} \\ &\Gamma_{1ii} = \begin{pmatrix} A_{ii}Q_{i1} + B_i X_i & Y_i C_i \\ 0 & A_{ii}Q_{i2} - Y_i C_i \end{pmatrix}, \\ &\Gamma_{1ij} = \begin{pmatrix} 0 & 0 \\ A_{ij}Q_{j1} & A_{ij}Q_{j2} \end{pmatrix}, \quad i \neq j \\ &\Gamma_{2i} = \begin{pmatrix} F_i \operatorname{diag}\left\{\begin{pmatrix} Q_{i1} & Q_{i2} \\ Q_{i1} & Q_{i2} \end{pmatrix}, \ldots, \begin{pmatrix} Q_{N1} & Q_{N2} \\ Q_{N1} & Q_{N2} \end{pmatrix}\right\} \\ H_i \operatorname{diag}\{(X_1 \quad 0), \ldots, (X_N \quad 0)\} \end{pmatrix}, \\ &i, j = 1, 2, \cdots, N \end{aligned}\right\} \tag{69}$$

In this case, a decentralized dynamic output feedback law can be given by (64) *with*

$$\begin{aligned} &L_i = Y_i U_i C_{0i} Q_{i21}^{-1} C_{0i}^{-1} U_i', \qquad K_i = X_i Q_{i1}^{-1}, \\ &i = 1, 2, \ldots, N \end{aligned} \tag{70}$$

Remark 12: The main results of § 5 extends those of §§ 3 and 4. From Remark 10, we can see that § 5 also extends the main result on decentralized static state feedback control design developed by Stipanovic and Siljak (2001).

6. Numerical examples

In the following three examples, MATLAB LMI Toolbox (Gahinet *et al*. 1995) is used to compute the convex optimization problem.

Example 1: Consider the discrete-time system (see Example 3 by Stipanovic and Siljak (2001)

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x(k) \\ &\quad + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + f(k, x(k), u(k)) \end{aligned} \tag{71}$$

with quadratic constraint (12), where $F = I_2$ and $H = 0$.

In this example, Corollary 1 is applied to construct a static state feedback law.

We solve the convex optimization problem (22) and obtain

$$\begin{aligned} &\alpha_{\max} = 0.6283, \qquad Q = \begin{pmatrix} 2.4780 & 0.3003 \\ 0.3003 & 1.0376 \end{pmatrix}, \\ &X = (5.8854 \quad 3.6984) \end{aligned} \tag{72}$$

Then by Corollary 1 the control gain for state feedback can be chosen as

$$K = (2.0137 \quad 2.9815). \tag{73}$$

It is worth pointing out that the maximal bound $\alpha_{\max} = 0.6283$ is larger than the bound 0.6015 given by Stipanovic and Siljak (2001). This is because the restriction on a special structure of L has to be chosen in LMI by Stipanovic and Siljak (2001) while there is no restriction on the choice of Q and X in Corollary 1 to make the bound less conservative. Also see Remark 5.

Example 2: Consider the discrete-time system

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -2 & 0 \end{pmatrix} x(k) \\ &\quad + \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & -1 \end{pmatrix} u(k) + f(k, x(k), u(k)), \\ y(k) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(k) \end{aligned} \tag{74}$$

with quadratic constraint (12), where $F = I_3$ and $H = I_2$.

The static output feedback in the form of (14) can be constructed by searching the solution of the convex optimization of (27). In this case, similar to Example 1, we have

$$\alpha_{\max} = 0.1107, \qquad Q_1 = \begin{pmatrix} 48.5971 & 4.1753 \\ 4.1753 & 2.2294 \end{pmatrix},$$
$$Q_2 = 1.0018, \qquad X = \begin{pmatrix} 32.9569 & -4.1744 \\ -8.3505 & -4.4584 \end{pmatrix} \tag{75}$$

Therefore we can get static output feedback as in the form of (14) with the gain

$$K = XQ_1^{-1} = \begin{pmatrix} 0.9999 & -3.7451 \\ -0.0000 & -1.9998 \end{pmatrix}$$

In addition, Theorem 4 can be used to construct a dynamic output feedback controller. Similarly, a solution of convex optimization problem (38) can be obtained as

$$\left.\begin{aligned} &\alpha_{\max} = 0.0463, \\ &Q_1 = \begin{pmatrix} 270.5528 & 36.6934 & 50.4699 \\ 36.6934 & 46.2278 & 82.9493 \\ 50.4699 & 82.9493 & 206.6651 \end{pmatrix} \\ &Q_{21} = \begin{pmatrix} 18.4924 & 7.1526 \\ 7.1526 & 4.3040 \end{pmatrix}, \\ &Q_{22} = 1.0034 \\ &X = \begin{pmatrix} 171.7812 & -47.6583 & -58.6581 \\ -82.4515 & -60.9956 & -124.6559 \end{pmatrix}, \\ &Y = \begin{pmatrix} 10.7965 & 2.5426 \\ -6.9913 & -0.2423 \\ -14.3045 & -8.6072 \end{pmatrix} \end{aligned}\right\} \tag{76}$$

Therefore a dynamic output feedback controller can be constructed as in the form of (30) with

$$L = \begin{pmatrix} 0.9948 & -1.0624 \\ -0.9974 & 1.6012 \\ -0.0001 & -1.9997 \end{pmatrix}, \tag{77}$$
$$K = \begin{pmatrix} 0.9132 & -3.0249 & 0.7072 \\ -0.1604 & -0.6437 & -0.3057 \end{pmatrix}$$

Example 3: Let us consider the following discrete-time interconnected system, which is the same as (46) by Stipanovic and Siljak (2001).

$$\left.\begin{aligned} x_1(k+1) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & 3 \end{pmatrix} x_1(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_1(k) + f_1(k, x(k)) \\ x_2(k+1) &= \begin{pmatrix} 0 & 1 \\ 4 & 5 \end{pmatrix} x_2(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2(k) + f_2(k, x(k)) \end{aligned}\right\} \tag{78}$$

with quadratic constraint (43), where $F_1 = I_3$, $F_2 = I_2$, $H_1 = 0$ and $H_2 = 0$.

From Corollary 2, we can solve the optimization problem (57)–(58) and obtain (after 25 iterations)

$$\alpha_1 = 0.3191, \qquad \alpha_2 = 0.3273 \tag{79}$$

with the control gains

$$\begin{aligned} K_1 &= (\,1.0000 \quad -4.0000 \quad -3.0000\,), \\ K_2 &= (\,-4.0000 \quad -5.0000\,) \end{aligned} \tag{80}$$

It is interesting to find that the maximal bounds in this example are larger than the bounds $\alpha_1 = \alpha_2 = 0.3132$ given by Stipanovic and Siljak (2001) with the same control gain as those in (80). The reason is similar to that given in Example 1. It shows that our approach is less conservative than that in Stipanovic and Siljak (2001).

7. Conclusion

This paper has studied the problems of robust stabilization by means of an output feedback law for a class of discrete-time system with non-linear perturbations. It is shown that the problems can be reformulated as convex optimization problems in the form of LMI. The sufficient conditions for existence of output feedback laws are obtained. Some extensions are made for a class of discrete-time non-linear decentralized system. The unified approach presented in this paper has improved and generalized the design techniques in the literature.

Acknowledgements

This work was supported by Hong Kong Research Grant Council (CityU 1138/01P) and partially supported by a grant (7001043) from City University of Hong Kong.

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