Ergodicity of Stochastic 2D Navier-Stokes equations with Lévy Noise

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Introduction

The Navier-Stokes equations are the fundamental model of the fluids. Despite their great physical importance, existence and uniqueness results for the equations in the three-dimensional case are still not known, and only the two-dimensional (2D in short) situation is amenable to a complete mathematical treatment. In the past years, many authors studied this equation in the random situations. Most of the works are with Gaussian white noise, e.g. ([2], [3], [8]-[13]) and references cited there. As we know, there are a few articles for the non Gaussian white noise, see [1] [4]-[7] [14] ets for the Lévy space-time white noise and Poisson random measure.

This talk is concerned with 2D Navier-Stokes equation with Lévy noise. The existence and uniqueness of the global strong and weak solutions and the existence of invariant measures is proved in [4]. But in that framework, it seems that it is impossible to get the strong Feller property. In the article [5], we prove the solution in a suitable state space, on which the solution is strong Feller. Our approach is based on the methods of [8]. For getting the ergodicity, the priori estimations and stopping time technique which were used in [5] play the key role in the proofs.

$(\Omega, \mathcal{F}, P)$: a complete probability space

$\{\mathcal{F}_t, t \geq 0\}$: an increasing and right continuous family of complete sub-σ-algebras

$N(ds, du)$: the Poisson measure with σ-finite intensity measure $n(du)$ on measurable space Z.

$\tilde{N}(ds, du) = N(ds, du) - n(du)ds$: the compensating martingale measure.

$W(t)$: the cylindrical Wiener process with covariance operator I. Q is a trace class.

Assume: $W(t)$ and $\tilde{N}(dt, du)$ are independent. $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$: the torus.

Consider the following stochastic equations in $\mathbb{T}^2$

$$\begin{cases} dX(t) = \left[\nu \triangle X(t) - (X(t)\cdot\nabla)\, X(t) - \nabla p(t)\right] dt \\ \qquad\qquad + \int_Z f(X(t-),u)\widetilde{N}(dt,du), \\ div\ X(t) = 0, \\ X(0) = x, \end{cases}$$

and

$$\begin{cases} dX(t) = \left[\nu \triangle X(t) - (X(t)\cdot\nabla)\, X(t) - p(t)\right] dt \\ \qquad\qquad + \int_Z f(X(t-),u)\widetilde{N}(dt,du) + \sqrt{Q}dW(t), \\ div\ X(t) = 0, \\ X(0) = x, \end{cases}$$

where $X(t)$ and $p(t)$ represent the velocity and pressure of the particle at time t, the positive parameter ν is the kinematic viscosity.

We consider a Hilbert space $\mathbb{H}$ which is a closed subspace of $L^2(\mathbb{T}^2, \mathbb{R}^2)$

$$\mathbb{H} = \left\{ u \in L^2(\mathbb{T}^2, \mathbb{R}^2), \mathrm{div} u = 0 \text{ and } \int_{\mathbb{T}^2} u(x)dx = 0 \right\}.$$

$$\mathbb{V} = \left\{ u \in H^1(\mathbb{T}^2, \mathbb{R}^2), \mathrm{div} u = 0 \text{ and } \int_{\mathbb{T}^2} u(x)dx = 0 \right\}$$

Let $\mathbb{H}^{-1}$ be the dual space of $\mathbb{H}^1$. The above two equations are equivalent in $\mathbb{H}^{-1}$ with the following

$$\begin{cases} dX(t) + [\nu AX(t) + (X(t) \cdot \nabla) X(t) - \nabla p(t)]dt \\ \qquad\qquad = \int_Z f(X(t-), u)\widetilde{N}(dt, du), \quad t > 0, \\ X(0) = x \end{cases} \tag{1}$$

and

$$\begin{cases} dX(t) + [\nu AX(t) + (X(t) \cdot \nabla) X(t) - \nabla p(t, \xi)]dt \\ \qquad\qquad = \int_Z f(X(t-), u)\widetilde{N}(dt, du) + \sqrt{Q}dW(t), \quad t > 0, \\ X(0) = x. \end{cases} \tag{2}$$

Taking the inner product of (1) and (2) with a function $v \in \mathbb{V}$ respectively, and integrating the second and the pressure term, we have

$$\begin{cases} \frac{d}{dt}\langle X(t), v\rangle + \nu\langle\langle X(t), v\rangle\rangle + b(X(t), X(t), v) \\ \qquad\qquad = \int_Z \langle f(X(t-), u), v\rangle \widetilde{N}(dt, du), \\ \langle X(0), v\rangle = \langle x, v\rangle \end{cases} \tag{3}$$

and

$$\begin{cases} \frac{d}{dt}\langle X(t), v\rangle + \nu\langle\langle X(t), v\rangle\rangle + b(X(t), X(t), v) \\ \qquad\qquad = \int_Z \langle f(X(t-), u), v\rangle \widetilde{N}(dt, du) + \left\langle \sqrt{Q} dW_t, v\right\rangle, \\ \langle X(0), v\rangle = \langle x, v\rangle \end{cases} \tag{4}$$

with

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathbb{T}^2} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx.$$

Define the bilinear operator $B(u,v) : \mathbb{V} \times \mathbb{V} \to \mathbb{V}^{-1}$,

$$\langle B(u,v), w \rangle = b(u,v,w), \quad Bu = B(u,u), \qquad u,v,w \in \mathbb{V}.$$

An alternative form of (3) and (4) can be rewrite as following:

$$\begin{cases} \frac{d}{dt}X(t) + \nu AX(t) + BX(t) = \int_Z f(X(t-),u)\widetilde{N}(dt,du), \\ X(0) = x \end{cases} \tag{5}$$

and

$$\begin{cases} \frac{d}{dt}X(t) + \nu AX(t) + BX(t) \\ \qquad\qquad = \int_Z f(X(t-),u)\widetilde{N}(dt,du) + \sqrt{Q}dW(t), \\ X(0) = x. \end{cases} \tag{6}$$

Definitions of weak solution

Suppose X(t) be right continuous with left limit in $\mathbb{H}$. If for $t > 0$, $\int_0^t [|X(s)|^2 + |BX(s)|^2] ds < \infty$ and for $v \in \mathcal{D}(A)$, and $\mathbb{P} - a.s.$

$$\langle X(t), v\rangle = \langle x, v\rangle - \nu \int_0^t \langle X(s), Av\rangle ds - \int_0^t \langle BX(s), v\rangle ds$$

$$\int_0^t \langle v, \sqrt{Q} dW(s)\rangle + \int_0^t \int_Z \langle f(X(s-), u), v\rangle \widetilde{N}(ds, du).$$

Fixed the measurable subset U_m of U with $U_m \uparrow U$ and $\lambda(U_m) < \infty$.

Hypothesis 1. There exists positive constants C, K such that

(1) $\int_Z \|f(0,u)\|^2 \lambda(du) = C < \infty$;

(2) $\int_Z \|f(x,u) - f(y,u)\|^2 \lambda(du) \leq K|x-y|^2$;

(3) $\sup_{|x|\leq M} \int_{U_k^c} |f(x,u)|^2 \lambda(du) \downarrow 0 \;\; \text{as}, \;\; k \uparrow \infty$,

Theorem 1. Suppose that **Hypothesis 1.** hold.
(i) For the initial value $x \in \mathbb{H}$, Eq.(1) and (2) has a unique global weak solution on $\mathbb{H}$.

(ii) There exists an invariant probability measure for X_t which is the solution of equation (1) and (2) on $\mathbb{H}$ which is loaded on $\mathbb{V}$.

Hypothesis 2. There exists positive constants C, K such that, for some $\alpha \in [1/4, 1/2)$, $\varepsilon > 0$

$(\mathbf{H_1})$ $Q : \mathbb{H} \to \mathbb{H}$ is a linear bounded operator, injective, with range $\mathscr{R}(Q)$ dense in $\mathscr{D}(A^{\frac{1}{4}+\frac{\alpha}{2}})$ and $\mathscr{D}(A^{2\alpha}) \subset \mathscr{R}(Q) \subset \mathscr{D}(A^{\frac{1}{4}+\frac{\alpha}{2}+\varepsilon})$;

$(\mathbf{H_2})$ $\int_U |A^\alpha f(0,u)|^2 \lambda(du) = C < \infty$;

$(\mathbf{H_3})$ $\int_U |A^\alpha (f(x,u) - f(y,u))|^2 \lambda(du) \leq K |A^\alpha (x-y)|^2$, $x, y \in \mathscr{D}(A^\alpha)$;

$(\mathbf{H_4})$ $\sup_{x \in \mathscr{D}(A^\alpha)} \int_{U_m^c} |A^\alpha f(x,u)|^2 \lambda(du) \to 0, \quad as \quad m \to \infty.$

Let z be the Ornstein-Uhlenbeck process that is the solution of

$$\begin{cases} dz_t + Az_t dt = QdW_t, \\ z_0 = 0. \end{cases}$$

Theorem 2. Suppose that **Hypothesis 2.** hold.
(i) For $x \in \mathscr{D}(A^\alpha)$, there exists a unique solution X of (1),(2) such that, for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$X - z \in D([0,T], \mathscr{D}(A^\alpha)) \cap L^{\frac{4}{1-2\alpha}}(0,T; \mathscr{D}(A^{\frac{1}{4}+\frac{\alpha}{2}}).$$

(ii) $(P_t)_{t\geq 0}$ of (2) is a strong Feller group on $C_b(\mathscr{D}(A^\alpha))$.
(iii)The solution X of (2) is irreducible on $\mathscr{D}(A^\alpha)$.

Theorem 3. Suppose that **Hypothesis 2.** hold. Then there is a unique invariant measure for X_t which is the solution of (2). Furthermore, the transition probability $P(t, x, \cdot)$ and $P(t, y, \cdot)$ of the X_t are absolutely continuous for different x, y in $\mathscr{D}(A^\alpha)$, and they are also absolutely continuous with the invariant probability on $\mathscr{D}(A^\alpha)$. Furthermore, X_t is ergodic on $\mathbb{H}$.

Outline of the proof the Theorem 1(i).

Step 1: Consider the case of $\lambda(Z) < \infty$.
There exist a global strong solution.

Step 2: Consider the case of $\lambda(Z) = \infty$.
Let U_k be a measurable subset of Z with $\lambda(U_k) < \infty$ and $U_k \uparrow Z$, then prove the solution X_k has a limit X, and X is the desired global strong solution.

Step 1: Since the character measure $\lambda(Z) < \infty$, we can arrange the jump times of $N(dt, du)$. Assume the jump times of $N(dt, du)$ are $\sigma_1(\omega) < \sigma_2(\omega) < \cdots$, each $\sigma_i, i = 1, 2, \cdots$ is stopping time. Since for $t \in [0, \sigma_1)$,

$$\int_0^t \int_Z f(X(s-, u)\tilde{N}(ds, du) = -\int_0^t \int_Z f(X(s, u)\lambda(du)ds.$$

the equation (5) is equivalent with the determined integral differential equation

$$\begin{cases} \dfrac{dX(t)}{dt} + \nu AX(t) + B(X(t)) = -\displaystyle\int_Z f(X(t), u)\lambda(du), \\ X(0) = x, \quad x \in L^2(\mathbb{T}^2). \end{cases} \tag{7}$$

We will show that there exist a weak solution of equation (7) by using the Galerkin methods.
Let $\mathbb{E}_n = \text{span}\{e_1, e_2, \cdots, e_n\}, (n = 1, 2, \cdots)$, P_n is an orthonormal projection from $L^2(\mathbb{T}^2, \mathbb{R}^2) \to \mathbb{E}_n$. Denote by $|\cdot|_{\mathbb{E}_n}$ and $\|\cdot\|_{\mathbb{E}_n}$ the norm of $\mathbb{H}$ and $\mathbb{H}^1$ on $\mathbb{E}_n$ respectively. Let $X_n(t) = \sum_{j=1}^n \langle X_n(t), e_j\rangle e_j$, the orthonormal projection of $X(t)$ on $\mathbb{E}_n$ and consider the following ordinary differential equation

$$\begin{cases} \dfrac{dX_n(t)}{dt} = \nu\Delta X_n(t) + P_n B X_n(t) - \displaystyle\int_Z P_n f(X_n(s), u)\lambda(du), \\ X_n(0) = P_n x. \end{cases} \tag{8}$$

Lemma 1 Equation (8) has a unique strong global solution in $C([0,T]; \mathcal{D}(\Delta)) \cap L^2(0,T;\mathbb{H})$.

Proof: **Existence.** Let $X_n(t)$ be the local solution of equation (8),
Take the inner product of with ΔX_n, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|X_n(t)\|^2 + \nu|\Delta X_n(t)|^2 + \langle P_n B X_n(t), \Delta X_n(t)\rangle \\ = & \int_Z \langle P_n f(X_n(t), u), \Delta X_n(t)\rangle \lambda(du). \end{aligned}$$

From the hypotheses of f, the Young inequality

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|X_n(t)\|^2 + \nu|\Delta X_n(t)|^2 \\ \leq & \frac{\nu}{2}|\Delta X_n(t)|^2 + \frac{2}{\nu}[(2\lambda(Z)K)^{1/2}|X_n(t)| + (2\lambda(Z)C)^{1/2}]^2, \end{aligned}$$

Integrate the inequality on both side from 0 to t, we have

$$\begin{aligned}&\|X_n(t)\|^2+\nu\int_0^t|\Delta X_n(s)|^2ds\\ \leq\ &\|x\|^2+\frac{2}{\nu}\int_0^t\Big[(2\lambda(Z)K)^{1/2}|X_n(s)|+(2\lambda(Z)C)^{1/2}\Big]^2ds.\end{aligned}$$

Thus X_n is uniformly bounded in $L^\infty(0,T;\mathbb{V})\cap L^2(0,T;\mathscr{D}(A))$. Furthermore we can prove that $\{dX_n(t)/dt\}_{n\geq 1}$ is uniformly bounded in $L^2(0,T;\mathbb{H})$.

By using the Alaoglu week compact Theorem , the Reflexive weak compactness theorem, we extract a sequence such that

$$
\begin{aligned}
X_n &\overset{*}{\rightharpoonup} X && \text{in} && L^\infty(0,T;\mathbb{V}),\\
X_n &\rightharpoonup X && \text{in} && L^2(0,T;\mathscr{D}(A)),\\
\frac{d}{dt}X_n &\rightharpoonup \frac{d}{dt}X && \text{in} && L^2(0,T;\mathbb{H}).
\end{aligned}
$$

for some $X \in L^\infty(0,T;\mathbb{V}) \cap L^2(0,T;\mathscr{D}(A))$.
It is easy to show that X is the strong solution of equation (8).

Uniqueness. Let X and Y be two solutions of (8). Set $U(t) = X(t) - Y(t)$ and consider

$$\frac{d}{dt}U(t)+\nu\Delta U(t)+BX(t)-BY(t) = \int_Z \left[f(Y(t),u) - f(X(t),u)\right]\lambda(du)$$

Taking the inner product with ΔU, by the Gronwall inequality

$$\begin{aligned}\|U(t)\|^2 \;\; \leq \;\; & \|U(0)\|^2 \exp\left\{\frac{2}{\lambda_1}\left[\frac{K_1^2\|X\|_{L^\infty(0,T;\mathbb{V})}}{\nu}\int_0^T |\Delta X(s)|ds\right.\right. \\ & + \left.\left.\frac{27K_1^4T\|Y\|^4_{L^\infty(0,T;\mathbb{V})}}{4\nu}+\frac{K\lambda(Z)T}{\nu}\right]\right\}.\end{aligned}$$

Step 2: $\lambda(Z) = \infty$.

Let U_n be a measurable subset of Z with $\lambda(U_n) < \infty$ and $U_n \uparrow Z$, we want to prove the solution X_n has a limit X, and X is the desired solution

For every $k \geq 1$, consider the equation

$$\begin{cases} dX^n(t) + \nu AX^n(t)dt + BX^n(t) = \int_{U_n} f(X^k(t-), u)\widetilde{N}(dt, du), t > 0, \\ X^n(0) = x, \end{cases} \tag{9}$$

By using Itô's formula,we have

$$\begin{aligned}|X_t^n|^2 = &\ |x|^2 - 2\nu\int_0^t \|X_s^n\|^2 ds + 2\int_0^t\int_{U_n} \langle X_{s-}^n, f(X_{s-}^n, u)\rangle \widetilde{N}(ds, du) \\ &+ \sum_{s\le t}\left[\int_{U_n} |f(X_{s-}^n, u)| N(\{s\}, du)\right]^2 \\ \le &\ |x|^2 - 2\nu\int_0^t \|X_s^n\|^2 ds + 2\int_0^t\int_{U_n} \langle X_{s-}^n, f(X_{s-}^n, u)\rangle \widetilde{N}(ds, du) \\ &+ \int_0^t\int_{U_n} |f(X_{s-}^n, u)|^2 N(ds, du). \end{aligned} \tag{10}$$

For any fixed $k \geq 1$ and $n \geq 1$, define the stoping time

$$\tau_k^n = \inf \left\{ t > 0 : |X_t^n|^2 \vee \int_0^t \|X_s^n\|^2 ds > k \right\},$$

By the Davis' inequality, Young inequality,Gronwall inequality

$$\begin{aligned}
&\mathbb{E} \sup_{s \leq \tau_k^n \wedge t} |X_s^n - X_s^m|^2 + \nu \mathbb{E} \int_0^{\tau_k^n \wedge t} \|X_s^n - X_s^m\|^2 ds \\
\leq &\left(\frac{KC(k,t)\varepsilon + KC^2(k,t)}{\varepsilon - 2\varepsilon^2} \right) \mathbb{E} \int_0^t \sup_{s' \leq \tau_k^n \wedge s} |X_{s'}^n - X_{s'}^m|^2 ds \\
&+ \left(\frac{C(k,t)\varepsilon + C^2(k,t)}{\varepsilon - 2\varepsilon^2} \right) \mathbb{E} \int_0^t \int_{U_m^c} |f(X_{\tau_k^n \wedge s}^n, u)|^2 \lambda(du) ds.
\end{aligned}$$

From Gronwall inequality,

$$
\begin{aligned}
& \mathbb{E} \sup_{s \leq \tau_k^n \wedge t} |X_s^n - X_s^m|^2 + \nu \mathbb{E} \int_0^{\tau_k^n \wedge t} \|X_s^n - X_s^m\|^2 ds \\
\leq \quad & C(k, K, \varepsilon) \mathbb{E} \int_0^t \int_{U_m^c} |f(X_{\tau_k^n \wedge s}^n, u)|^2 \lambda(du) ds \\
\leq \quad & C(k, K, \varepsilon) \sup_{|x| \leq \sqrt{k}} \int_{U_m^c} |f(x, u)|^2 \lambda(du).
\end{aligned}
$$

From $(\mathbf{H_3})$

$$
\lim_{m \to \infty} \left[\mathbb{E} \sup_{s \leq \tau_k^n \wedge t} |X_s^n - X_s^m|^2 + \nu \mathbb{E} \int_0^{\tau_k^n \wedge t} \|X_s^n - X_s^m\|^2 ds \right] = 0.
$$

Since we have have

$$\mathbb{P}(t > \tau_k^n) = \mathbb{P}\left(\sup_{s\le t}|X_t^n|^2 \vee \int_0^t \|X_s^n\|^2 \ge k\right) \le \frac{C(t)}{k}$$

for some positive constant $C(t)$. Then

$$\begin{aligned}
&\mathbb{E}\sup_{s\le t}|X_s^n - X_s^m| \\
= &\ \mathbb{E}\sup_{s\le t}|X_s^n - X_s^m| I_{\{t\le\tau_k^n\}} + \mathbb{E}\sup_{s\le t}|X_s^n - X_s^m| I_{\{t>\tau_k^n\}} \\
\le &\ \mathbb{E}\sup_{s\le\tau_k^n\wedge t}|X_s^n - X_s^m| + \left(\mathbb{E}\sup_{s\le t}|X_s^n - X_s^m|^2\right)^{1/2}[\mathbb{P}(t > \tau_k^n)]^{1/2} \\
\le &\ \left(\mathbb{E}\sup_{s\le\tau_k^n\wedge t}|X_s^n - X_s^m|^2\right)^{1/2} + C(t)\left(\frac{C(t)}{k}\right)^{1/2}.
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\int_0^t \|X_s^n - X_s^m\| ds I_{\{t\le\tau_k^n\}} + \mathbb{E}\int_0^t \|X_s^n - X_s^m\| ds I_{\{t>\tau_k^n\}} \\
\le\ & \mathbb{E}\int_0^{\tau_k^n\wedge t} \|X_s^n - X_s^m\| ds I_{\{t\le\tau_k^n\}} + \mathbb{E}\int_0^t \|X_s^n - X_s^m\| ds I_{\{t>\tau_k^n\}} \\
\le\ & \mathbb{E}\int_0^{\tau_k^n\wedge t} \|X_s^n - X_s^m\| ds + tC(t)\mathbb{P}^{\frac{1}{2}}(t>\tau_k^n) \\
\le\ & t\mathbb{E}\int_0^{\tau_k^n\wedge t} \|X_s^n - X_s^m\|^2 ds + tC(t)\left(\frac{C(t)}{k}\right)^{1/2}.
\end{aligned}
$$

we have that, for any fixed t,

$$
\lim_{m\to\infty}\left[\mathbb{E}\sup_{s\le t}|X_s^n - X_s^m| + \mathbb{E}\int_0^t \|X_s^n - X_s^m\| ds\right] = 0.
$$

This means that, $\{X^n\}_{n\geq 1}$ is a Cauchy sequence in the space $\widetilde{\mathscr{P}}_T$ which is the space of all $\mathbb{H}$-valued adapted càdlàg process with $\mathbb{E}\left(\sup_{t\leq T}|X_t| + \int_0^T \|X_s\| ds\right) < \infty$ for any positive number T.

Hence, there exists a process $X \in \widetilde{\mathscr{P}}_T$ such that

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{t\leq T}|X_t^n - X_t| + \int_0^T \|X_s^n - X_s\| \, ds\right) = 0.$$

We can prove that X is a weak solution of (1.5).

Outline of the proof the Theorem 1(ii).
By Ito formula

$$
\begin{aligned}
& |X_t|^2 \\
= & |x|^2 - 2\nu \int_0^t \|X_s\|^2 ds - 2 \int_0^t b(X_s, X_s, X_s) ds \\
& +2 \int_0^t \int_Z \langle X_{s-}, f(X_{s-}, u) \rangle \widetilde{N}(ds, du) + \sum_{s \le t} \left| \int_Z f(X_{s-}, u) N(\{s\}, du) \right| \\
\le & |x|^2 - 2\nu \int_0^t \|X_s\|^2 ds + 2 \int_0^t \int_Z \langle X_{s-}, f(X_{s-}, u) \rangle \widetilde{N}(ds, du) \\
& + \int_0^t \int_Z |f(X_{s-}, u)|^2 N(ds, du).
\end{aligned}
$$

From this inequality and the assumption of the proposition, we get

$$2\nu \int_0^t \mathbb{E}\|X_s\|^2 ds \leq \ (|x|^2 + 4Ct) + \frac{2K(|x|^2 + 2Ct)}{\nu\lambda_1 - K}.$$

Let $P(t, x, A)$ be the transition probability measure of X and define

$$\mu_T(A) = \frac{1}{T} \int_0^T P(t, x, A) dt.$$

For $R > 0$, let $B_R = \{x \in \mathbb{H}; \|x\| \leq R\}$, we have

$$\mu_T(B_R^c) = \frac{1}{T} \int_0^T P(t, x, B_R^c) dt \leq \frac{1}{TR^2} \int_0^T \mathbb{E}\|X_t\|^2 dt \leq \frac{M}{R^2}$$

for some constant $M > 0$. Hence $\{\mu_T, T > 0\}$ is tight and its limit is an invariant probability measure of the solution X.

Strong Feller + irreducibility

Lemma 1. Suppose that Hypothesis 2. holds. This is a Markov process of (2) satisfying the Feller property on $\mathscr{D}(A^\alpha)$.

Lemma 2. Suppose that Hypothesis 2. holds. The solution of (2) is irreducible on $\mathscr{D}(A^\alpha)$.

Outline of prove Lemma 1.

First prove the solution X^R of the truncated equation (2) has strong Feller property, then take limitation.

we get the following Bismut-Elworthy-Li formula for stochastic 2D Navier-Stokes equation with jumps:

$$D_x\mathbb{E}\varphi(X_t(x))\cdot h=\frac{1}{t}\mathbb{E}\left[\varphi(X_t(x))\int_0^t\left\langle(QQ^*)^{-1/2}\eta_s^h(x),dW_s\right\rangle\right]. \tag{11}$$

where $\eta_t^h(x)$ is the solution of the equation:

$$\begin{cases} d\eta_t^h(x)=-\left\{A\eta_t^h(x)+\left[\left(B(X_t(x),\eta_t^h(x)\right)+\left(B(\eta_t^h(x),X_t(x)\right)\right]\right\}dt \\ \qquad\qquad +\int_U Df(X_t(x),z)\cdot\eta_t^h(x)\widetilde{N}(dt,dz), \\ \eta_0^h(\xi)=h(\xi). \end{cases} \tag{12}$$

Set

$$\tau^k = \left\{ t > 0 : \int_0^t \|z_s\|_{L^4}^4 ds > k \right\}.$$

Since from the Hypothesis 2, we have For $k \geq 1$,

$$|(QQ^*)^{-1/2} y|^2 \leq C_1 |A^{2\alpha} y|^2 \qquad \forall y \in \mathscr{D}(A^{2\alpha}).$$

Therefore, we have, for any $x, y \in \mathscr{D}(A^\alpha)$ and $\varphi \in C_b(\mathscr{D}(A^\alpha))$,

$$\begin{aligned}
&|P_{t\wedge\tau^k}\varphi(x) - P_{t\wedge\tau^k}\varphi(y)| \\
\leq& \frac{C_1}{t} \|\varphi\|_\infty \sup_{\substack{k,h \in \mathscr{D}(A^\alpha) \\ |A^\alpha h| \leq 1}} \left[\mathbb{E} \int_0^t \left| A^{2\alpha} D_x X_{s\wedge\tau_k}^{(R)}(k) \cdot h \right|^2 ds \right]^{1/2} \|x - y\|_{\mathscr{D}(A^\alpha)} \\
\leq& \frac{C_1(R, k, \varepsilon, \lambda_1, t)}{t} \|\varphi\|_\infty \|x - y\|_{\mathscr{D}(A^\alpha)}.
\end{aligned}$$

Since $\tau_k \uparrow \infty, k \to \infty$, This implies that P_t is strong Feller on $\mathscr{D}(A^\alpha)$.

Outline of prove Lemma 2.

(i) For every $k \geq 1$, we firstly prove that the equation

$$\begin{aligned} dX^n(t) + \nu AX^n(t)dt + BX^n(t) &= \int_{U_n} f(X^k(t-), u)\widetilde{N}(dt, du) \\ &\quad + \sqrt{Q}dW_t \end{aligned}$$

is irreducible.

(ii) Let the transition probability of $X^n(t)$ is $P_t^0(x, C)$ and $P_t^f(x, C)$ be the above equation

$$
\begin{aligned}
P_t^f(x,C) = \; & e^{-t\lambda(U)} P_t^0(x,C) \\
& + \int_0^t \int_{\mathbb{H}} \int_U e^{-s\lambda(U)} P_{t-s}^f(y+f(y,z),C)\lambda(dz) P_s^0(x,dy).
\end{aligned}
$$

Then we can prove the solution of the second equation is irreducible.
(iii) Lastly for $y \in \mathscr{D}(A^\alpha)$,

$$
\begin{aligned}
& \mathbb{P}(\{\|X_t(x) - y\|_{\mathscr{D}(A^\alpha)} \geq 2\varepsilon\} \\
& \leq \mathbb{P}\{\|X_t(x) - X_t^n(x)\|_{\mathscr{D}(A^\alpha)} \geq \varepsilon\} + \mathbb{P}\{\|X_t^n(x) - y\|_{\mathscr{D}(A^\alpha)} \geq \varepsilon\} \\
& < 1.
\end{aligned}
$$

Then $X_t(x)$ is irreducible.

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Thank You !