Ergodicity of Stochastic 2D Navier-Stokes equations with Lévy Noise

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Introduction

The existence and uniqueness of solution for the Navier-Stokes equation with Lévy Noise

The Ergodicity
The Navier-Stokes equations are the fundamental model of the fluids. Despite their great physical importance, existence and uniqueness results for the equations in the three-dimensional case are still not known, and only the two-dimensional (2D in short) situation is amenable to a complete mathematical treatment. In the past years, many authors studied this equation in the random situations. Most of the works are with Gaussian white noise, e.g. ([2], [3], [8]-[13]) and references cited there. As we know, there are a few articles for the non Gaussian white noise, see [1] [4]-[7] [14] etc for the Lévy space-time white noise and Poisson random measure.
This talk is concerned with 2D Navier-Stokes equation with Lévy noise. The existence and uniqueness of the global strong and weak solutions and the existence of invariant measures is proved in [4]. But in that framework, it seems that it is impossible to get the strong Feller property. In the article [5], we prove the solution in a suitable state space, on which the solution is strong Feller. Our approach is based on the methods of [8]. For getting the ergodicity, the priori estimations and stopping time technique which were used in [5] play the key role in the proofs.
\((\Omega, \mathcal{F}, P)\) : a complete probability space

\(\{\mathcal{F}_t, t \geq 0\} : \) an increasing and right continuous family of complete sub-\(\sigma\)-algebras

\(N(ds, du)\) : the Poisson measure with \(\sigma\)-finite intensity measure \(n(du)\) on measurable space \(Z\).

\(\tilde{N}(ds, du) = N(ds, du) - n(du)ds\) : the compensating martingale measure.

\(W(t)\) : the cylindrical Wiener process with covariance operator \(I\).

\(Q\) is a trace class.

Assume: \(W(t)\) and \(\tilde{N}(dt, du)\) are independent. \(\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2\) : the torus.
Consider the following stochastic equations in $\mathbb{T}^2$

\[
\begin{aligned}
dX(t) &= \left[ \nu \Delta X(t) - (X(t) \cdot \nabla) X(t) - \nabla p(t) \right] dt \\
&\quad + \int_{\mathbb{Z}} f(X(t-), u) \tilde{N}(dt, du), \\
div X(t) &= 0, \\
X(0) &= x,
\end{aligned}
\]

and

\[
\begin{aligned}
dX(t) &= \left[ \nu \Delta X(t) - (X(t) \cdot \nabla) X(t) - p(t) \right] dt \\
&\quad + \int_{\mathbb{Z}} f(X(t-), u) \tilde{N}(dt, du) + \sqrt{Q}dW(t), \\
div X(t) &= 0, \\
X(0) &= x,
\end{aligned}
\]

where $X(t)$ and $p(t)$ represent the velocity and pressure of the particle at time $t$, the positive parameter $\nu$ is the kinematic viscosity.
We consider a Hilbert space $\mathbb{H}$ which is a closed subspace of $L^2(\mathbb{T}^2, \mathbb{R}^2)$

\[
\mathbb{H} = \left\{ u \in L^2(\mathbb{T}^2, \mathbb{R}^2), \text{div}u = 0 \text{ and } \int_{\mathbb{T}^2} u(x)dx = 0 \right\}.
\]

\[
\mathbb{V} = \left\{ u \in H^1(\mathbb{T}^2, \mathbb{R}^2), \text{div}u = 0 \text{ and } \int_{\mathbb{T}^2} u(x)dx = 0 \right\}
\]

Let $\mathbb{H}^{-1}$ be the dual space of $\mathbb{H}^1$. The above two equations are equivalent in $\mathbb{H}^{-1}$ with the following
\[
\begin{aligned}
\left\{
\begin{array}{l}
\frac{dX(t)}{dt} + [\nu A X(t) + (X(t) \cdot \nabla) X(t) - \nabla p(t)] dt \\
= \int_Z f(X(t-), u) \tilde{N}(dt, du), \quad t > 0,
\end{array}
\right.
\end{aligned}
\]
\begin{equation}
X(0) = x
\end{equation}

and
\[
\begin{aligned}
\left\{
\begin{array}{l}
\frac{dX(t)}{dt} + [\nu A X(t) + (X(t) \cdot \nabla) X(t) - \nabla p(t, \xi)] dt \\
= \int_Z f(X(t-), u) \tilde{N}(dt, du) + \sqrt{Q} dW(t), \quad t > 0,
\end{array}
\right.
\end{aligned}
\]
\begin{equation}
X(0) = x.
\end{equation}
Taking the inner product of (1) and (2) with a function $v \in \mathcal{V}$ respectively, and integrating the second and the pressure term, we have

$$
\begin{align*}
\frac{d}{dt} \langle X(t), v \rangle + \nu \langle \langle X(t), v \rangle \rangle + b(X(t), X(t), v) &= \int_Z \langle f(X(t-), u), v \rangle \tilde{N}(dt, du), \\
\langle X(0), v \rangle &= \langle x, v \rangle
\end{align*}
$$

(3)

and

$$
\begin{align*}
\frac{d}{dt} \langle X(t), v \rangle + \nu \langle \langle X(t), v \rangle \rangle + b(X(t), X(t), v) &= \int_Z \langle f(X(t-), u), v \rangle \tilde{N}(dt, du) + \langle \sqrt{Q}dW_t, v \rangle, \\
\langle X(0), v \rangle &= \langle x, v \rangle
\end{align*}
$$

(4)

with

$$
b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathbb{T}^2} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx.
$$
Define the bilinear operator $B(u, v) : \mathbb{V} \times \mathbb{V} \to \mathbb{V}^{-1}$,

$$\langle B(u, v), w \rangle = b(u, v, w), \quad Bu = B(u, u), \quad u, v, w \in \mathbb{V}.$$  

An alternative form of (3) and (4) can be rewrite as following:

$$\begin{cases} 
\frac{d}{dt}X(t) + \nu AX(t) + BX(t) = \int_{\mathbb{Z}} f(X(t-), u) \tilde{N}(dt, du), \\
X(0) = x
\end{cases}$$

(5)

and

$$\begin{cases} 
\frac{d}{dt}X(t) + \nu AX(t) + BX(t) \\
\quad = \int_{\mathbb{Z}} f(X(t-), u) \tilde{N}(dt, du) + \sqrt{Q}dW(t), \\
X(0) = x.
\end{cases}$$

(6)
Definitions of weak solution

Suppose $X(t)$ be right continuous with left limit in $\mathbb{H}$. If for $t > 0$, 
$$
\int_0^t [ |X(s)|^2 + |BX(s)|^2 ] ds < \infty
$$
and for $v \in D(A)$, and $\mathbb{P} – a.s.$

\[
\langle X(t), v \rangle = \langle x, v \rangle - \nu \int_0^t \langle X(s), Av \rangle ds - \int_0^t \langle BX(s), v \rangle ds \\
\int_0^t \langle v, \sqrt{Q}dW(s) \rangle + \int_0^t \int_Z \langle f(X(s-), u), v \rangle \tilde{N}(ds, du).
\]
Fixed the measurable subset $U_m$ of $U$ with $U_m \uparrow U$ and
\[ \lambda(U_m) < \infty. \]

**Hypothesis 1.** There exists positive constants $C, K$ such that
\begin{enumerate}
  \item \[ \int_Z \| f(0, u) \|^2 \lambda(du) = C < \infty; \]
  \item \[ \int_Z \| f(x, u) - f(y, u) \|^2 \lambda(du) \leq K |x - y|^2; \]
  \item \[ \sup_{|x| \leq M} \int_{U_k^c} |f(x, u)|^2 \lambda(du) \downarrow 0 \quad \text{as, } \quad k \uparrow \infty, \]
\end{enumerate}
Theorem 1. Suppose that Hypothesis 1. hold.
(i) For the initial value $x \in \mathbb{H}$, Eq.(1) and (2) has a unique global weak solution on $\mathbb{H}$.

(ii) There exists an invariant probability measure for $X_t$ which is the solution of equation (1) and (2) on $\mathbb{H}$ which is loaded on $\mathbb{V}$. 
Hypothesis 2. There exists positive constants $C, K$ such that, for some $\alpha \in [1/4, 1/2)$, $\varepsilon > 0$

$(H_1)$ $Q : \mathbb{H} \to \mathbb{H}$ is a linear bounded operator, injective, with range $\mathcal{R}(Q)$ dense in $\mathcal{D}(A^{1/4 + \alpha/2})$ and $\mathcal{D}(A^{2\alpha}) \subset \mathcal{R}(Q) \subset \mathcal{D}(A^{1/4 + \alpha/2 + \varepsilon})$;

$(H_2)$ $\int_{U} |A^{\alpha}f(0, u)|^2 \lambda(du) = C < \infty$;

$(H_3)$ $\int_{U} |A^{\alpha}(f(x, u) - f(y, u))|^2 \lambda(du) \leq K|A^{\alpha}(x - y)|^2,$

$x, y \in \mathcal{D}(A^{\alpha})$;

$(H_4)$ $\sup_{x \in \mathcal{D}(A^{\alpha})} \int_{U_m^c} |A^{\alpha}f(x, u)|^2 \lambda(du) \to 0, \text{ as } m \to \infty.$
Let \( z \) be the Ornstein-Uhlenbeck process that is the solution of
\[
\begin{aligned}
\left\{ \begin{array}{l}
\, dz_t + A z_t dt = Q dW_t, \\
\, z_0 = 0.
\end{array} \right.
\end{aligned}
\]

**Theorem 2.** Suppose that Hypothesis 2. hold.

(i) For \( x \in \mathcal{D}(A^{\alpha}) \), there exists a unique solution \( X \) of (1),(2) such that, for \( \mathbb{P} \)-a.s. \( \omega \in \Omega \),
\[
X - z \in D([0, T], \mathcal{D}(A^{\alpha})) \cap L^{\frac{4}{1-2\alpha}}(0, T; \mathcal{D}(A^{\frac{1}{4} + \frac{\alpha}{2}})).
\]

(ii) \((P_t)_{t \geq 0}\) of (2) is a strong Feller group on \( C_b(\mathcal{D}(A^{\alpha})) \).

(iii) The solution \( X \) of (2) is irreducible on \( \mathcal{D}(A^{\alpha}) \).
Theorem 3. Suppose that Hypothesis 2. hold. Then there is a unique invariant measure for $X_t$ which is the solution of (2). Furthermore, the transition probability $P(t, x, \cdot)$ and $P(t, y, \cdot)$ of the $X_t$ are absolutely continuous for different $x, y$ in $\mathcal{D}(A^\alpha)$, and they are also absolutely continuous with the invariant probability on $\mathcal{D}(A^\alpha)$. Furthermore, $X_t$ is ergodic on $\mathbb{H}$. 

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Outline of the proof the Theorem 1(i).

**Step 1:** Consider the case of $\lambda(Z) < \infty$.
There exist a global strong solution.

**Step 2:** Consider the case of $\lambda(Z) = \infty$.
Let $U_k$ be a measurable subset of $Z$ with $\lambda(U_k) < \infty$ and $U_k \uparrow Z$, then prove the solution $X_k$ has a limit $X$, and $X$ is the desired global strong solution.
**Step 1:** Since the character measure $\lambda(Z) < \infty$, we can arrange the jump times of $N(dt, du)$. Assume the jump times of $N(dt, du)$ are $\sigma_1(\omega) < \sigma_2(\omega) < \cdots$, each $\sigma_i, i = 1, 2, \cdots$ is stopping time. Since for $t \in [0, \sigma_1)$,

$$\int_0^t \int_Z f(X(s-, u)\tilde{N}(ds, du) = -\int_0^t \int_Z f(X(s, u)\lambda(du)ds.$$ 

the equation (5) is equivalent with the determined integral differential equation

$$\begin{aligned}
\frac{dX(t)}{dt} + \nu AX(t) + B(X(t)) &= -\int_Z f(X(t), u)\lambda(du), \\
X(0) &= x, \quad x \in L^2(\mathbb{T}^2).
\end{aligned}$$

(7)
We will show that there exist a weak solution of equation (7) by using the Galerkin methods.

Let $E_n = \text{span}\{e_1, e_2, \cdots, e_n\}, \ (n = 1, 2, \cdots)$, $P_n$ is an orthonormal projection from $L^2(\mathbb{T}^2, \mathbb{R}^2) \to E_n$. Denote by $| \cdot |_{E_n}$ and $\| \cdot \|_{E_n}$ the norm of $H$ and $H^1$ on $E_n$ respectively. Let $X_n(t) = \sum_{j=1}^{n} \langle X_n(t), e_j \rangle e_j$, the orthonormal projection of $X(t)$ on $E_n$ and consider the following ordinary differential equation

\[
\begin{cases}
\frac{dX_n(t)}{dt} = \nu \Delta X_n(t) + P_n BX_n(t) - \int_{Z} P_n f(X_n(s), u) \lambda(du), \\
X_n(0) = P_n x.
\end{cases}
\]

(8)

\textbf{Lemma 1} Equation (8) has a unique strong global solution in $C([0, T]; D(\Delta)) \cap L^2(0, T; H)$. 
Proof: Existence. Let $X_n(t)$ be the local solution of equation (8).
Take the inner product of with $\Delta X_n$, we have

$$
\frac{1}{2} \frac{d}{dt} \| X_n(t) \|^2 + \nu |\Delta X_n(t)|^2 + \langle P_n BX_n(t), \Delta X_n(t) \rangle \\
= \int_Z \langle P_n f(X_n(t), u), \Delta X_n(t) \rangle \lambda(du).
$$

From the hypotheses of $f$, the Young inequality

$$
\frac{1}{2} \frac{d}{dt} \| X_n(t) \|^2 + \nu |\Delta X_n(t)|^2 \\
\leq \nu |\Delta X_n(t)|^2 + \frac{2}{\nu} \left[ (2\lambda(Z)K)^{1/2} |X_n(t)| + (2\lambda(Z)C)^{1/2} \right]^2,
$$
Integrate the inequality on both side from 0 to t, we have

\[ \| X_n(t) \|^2 + \nu \int_0^t |\Delta X_n(s)|^2 \, ds \leq \| x \|^2 + \frac{2}{\nu} \int_0^t \left[ (2\lambda(Z)K)^{1/2} |X_n(s)| + (2\lambda(Z)C)^{1/2} \right]^2 \, ds. \]

Thus \( X_n \) is uniformly bounded in \( L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; \mathcal{D}(A)) \). Furthermore we can prove that \( \{dX_n(t)/dt\}_{n \geq 1} \) is uniformly bounded in \( L^2(0, T; \mathbb{H}) \).
By using the Alaoglu week compact Theorem, the Reflexive weak compactness theorem, we extract a sequence such that

\[ X_n \rightharpoonup^* X \quad \text{in} \quad L^\infty(0, T; \mathbb{V}), \]
\[ X_n \rightharpoonup X \quad \text{in} \quad L^2(0, T; \mathcal{D}(A)), \]
\[ \frac{d}{dt} X_n \rightharpoonup \frac{d}{dt} X \quad \text{in} \quad L^2(0, T; \mathbb{H}). \]

for some \( X \in L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; \mathcal{D}(A)). \)

It is easy to show that \( X \) is the strong solution of equation (8).
**Uniqueness.** Let $X$ and $Y$ be two solutions of (8). Set $U(t) = X(t) - Y(t)$ and consider

$$
\frac{d}{dt} U(t) + \nu \Delta U(t) + BX(t) - BY(t) = \int_Z \left[ f(Y(t), u) - f(X(t), u) \right] \lambda(du)
$$

Taking the inner product with $\Delta U$, by the Gronwall inequality

$$
\|U(t)\|^2 \leq \|U(0)\|^2 \exp \left\{ \frac{2}{\lambda_1} \left[ \frac{K^2_1 \|X\|_{L^\infty(0,T;\mathbb{V})}}{\nu} \int_0^T |\Delta X(s)| ds + \frac{27K^4_1 T \|Y\|_{L^\infty(0,T;\mathbb{V})}^4}{4\nu} + \frac{K \lambda(Z) T}{\nu} \right] \right\}.
$$
Step 2: $\lambda(Z) = \infty$.

Let $U_n$ be a measurable subset of $Z$ with $\lambda(U_n) < \infty$ and $U_n \uparrow Z$, we want to prove the solution $X_n$ has a limit $X$, and $X$ is the desired solution.

For every $k \geq 1$, consider the equation

\[
\begin{cases}
    dX^n(t) + \nu AX^n(t)dt + BX^n(t) = \int_{U_n} f(X^k(t^-), u)\tilde{N}(dt, du), t > 0, \\
    X^n(0) = x,
\end{cases}
\]

(9)
By using Itô’s formula, we have

\[
|X_t^n|^2 = |x|^2 - 2\nu \int_0^t \|X_s^n\|^2 ds + 2 \int_0^t \int_{U_n} \langle X_{s-}^n, f(X_{s-}^n, u) \rangle \tilde{N}(ds, du) \]

\[
+ \sum_{s \leq t} \left[ \int_{U_n} |f(X_{s-}^n, u)| N(\{s\}, du) \right]^2
\]

\[
\leq |x|^2 - 2\nu \int_0^t \|X_s^n\|^2 ds + 2 \int_0^t \int_{U_n} \langle X_{s-}^n, f(X_{s-}^n, u) \rangle \tilde{N}(ds, du) \]

\[
+ \int_0^t \int_{U_n} |f(X_{s-}^n, u)|^2 N(ds, du). \tag{10}
\]
For any fixed $k \geq 1$ and $n \geq 1$, define the stopping time

$$\tau_{k}^{n} = \inf \left\{ t > 0 : |X_{t}^{n}|^{2} \vee \int_{0}^{t} \|X_{s}^{n}\|^{2} ds > k \right\},$$

By the Davis’ inequality, Young inequality, Gronwall inequality

$$\mathbb{E} \sup_{s \leq \tau_{k}^{n} \wedge t} |X_{s}^{n} - X_{s}^{m}|^{2} + \nu \mathbb{E} \int_{0}^{\tau_{k}^{n} \wedge t} \|X_{s}^{n} - X_{s}^{m}\|^{2} ds \leq \left( \frac{KC(k, t)\varepsilon + KC^{2}(k, t)}{\varepsilon - 2\varepsilon^{2}} \right) \mathbb{E} \int_{0}^{t} \sup_{s' \leq \tau_{k}^{n} \wedge s} |X_{s'}^{n} - X_{s'}^{m}|^{2} ds$$

$$+ \left( \frac{C(k, t)\varepsilon + C^{2}(k, t)}{\varepsilon - 2\varepsilon^{2}} \right) \mathbb{E} \int_{0}^{t} \int_{U_{m}} |f(X_{\tau_{k}^{n} \wedge s}^{n}, u)|^{2} \lambda(du) ds.$$
From Gronwall inequality,

\[
\mathbb{E} \sup_{s \leq \tau_k \wedge t} |X^n_s - X^m_s|^2 + \nu \mathbb{E} \int_0^{\tau_k \wedge t} \|X^n_s - X^m_s\|^2 ds \\
\leq C(k, K, \varepsilon) \mathbb{E} \int_0^t \int_{U_m^c} |f(X^n_{\tau_k \wedge s}, u)|^2 \lambda(du) ds \\
\leq C(k, K, \varepsilon) \sup_{|x| \leq \sqrt{k}} \int_{U_m^c} |f(x, u)|^2 \lambda(du).
\]

From (H3)

\[
\lim_{m \to \infty} \left[ \mathbb{E} \sup_{s \leq \tau_k \wedge t} |X^n_s - X^m_s|^2 + \nu \mathbb{E} \int_0^{\tau_k \wedge t} \|X^n_s - X^m_s\|^2 ds \right] = 0.
\]
Since we have have
\[
\mathbb{P}(t > \tau^n_k) = \mathbb{P}\left( \sup_{s \leq t} |X^n_s|^2 \vee \int_0^t \|X^n_s\|^2 \geq k \right) \leq \frac{C(t)}{k}
\]
for some positive constant $C(t)$. Then
\[
\mathbb{E} \sup_{s \leq t} |X^n_s - X^m_s|
= \mathbb{E} \sup_{s \leq t} |X^n_s - X^m_s| I\{t \leq \tau^n_k\} + \mathbb{E} \sup_{s \leq t} |X^n_s - X^m_s| I\{t > \tau^n_k\}
\leq \mathbb{E} \sup_{s \leq \tau^n_k \wedge t} |X^n_s - X^m_s| + \left( \mathbb{E} \sup_{s \leq t} |X^n_s - X^m_s|^2 \right)^{1/2} \left[ \mathbb{P}(t > \tau^n_k) \right]^{1/2}
\leq \left( \mathbb{E} \sup_{s \leq \tau^n_k \wedge t} |X^n_s - X^m_s|^2 \right)^{1/2} + C(t) \left( \frac{C(t)}{k} \right)^{1/2}.
\]
\[
\begin{align*}
\mathbb{E} \int_0^t \|X^n_s - X^m_s\| ds I\{t \leq \tau^n_k\} + \mathbb{E} \int_0^t \|X^n_s - X^m_s\| ds I\{t > \tau^n_k\} & \\
\leq \mathbb{E} \int_0^{\tau^n_k \land t} \|X^n_s - X^m_s\| ds I\{t \leq \tau^n_k\} + \mathbb{E} \int_0^t \|X^n_s - X^m_s\| ds I\{t > \tau^n_k\} & \\
\leq \mathbb{E} \int_0^{\tau^n_k \land t} \|X^n_s - X^m_s\| ds + tC(t)\mathbb{P}^{1/2}(t > \tau^n_k) & \\
\leq t\mathbb{E} \int_0^{\tau^n_k \land t} \|X^n_s - X^m_s\|^2 ds + tC(t) \left(\frac{C(t)}{k}\right)^{1/2}.
\end{align*}
\]

we have that, for any fixed \(t\),

\[
\lim_{m \to \infty} \left[ \mathbb{E} \sup_{s \leq t} |X^n_s - X^m_s| + \mathbb{E} \int_0^t \|X^n_s - X^m_s\| ds \right] = 0.
\]
This means that, $\{X^n\}_{n \geq 1}$ is a Cauchy sequence in the space $\widetilde{P}_T$ which is the space of all $\mathbb{H}$-valued adapted càdlàg process with $\mathbb{E}\left(\sup_{t \leq T} |X_t| + \int_0^T \|X_s\| \, ds\right) < \infty$ for any positive number $T$. Hence, there exists a process $X \in \widetilde{P}_T$ such that

$$\lim_{n \to \infty} \mathbb{E}\left(\sup_{t \leq T} |X^n_t - X_t| + \int_0^T \|X^n_s - X_s\| \, ds\right) = 0.$$ 

We can prove that $X$ is a weak solution of (1.5).
Outline of the proof the Theorem 1(ii).

By Ito formula

\[|X_t|^2\]

\[= |x|^2 - 2\nu \int_0^t \|X_s\|^2 ds - 2 \int_0^t b(X_s, X_s, X_s) ds\]

\[+ 2 \int_0^t \int_Z \langle X_s-, f(X_s-, u) \rangle \tilde{N}(ds, du) + \sum_{s \leq t} \left| \int_Z f(X_s-, u) N(\{ s \}, du) \right|\]

\[\leq |x|^2 - 2\nu \int_0^t \|X_s\|^2 ds + 2 \int_0^t \int_Z \langle X_s-, f(X_s-, u) \rangle \tilde{N}(ds, du)\]

\[+ \int_0^t \int_Z |f(X_s-, u)|^2 N(ds, du).\]
From this inequality and the assumption of the proposition, we get

\[ 2\nu \int_0^t \mathbb{E}\|X_s\|^2 \, ds \leq (|x|^2 + 4Ct) + \frac{2K(|x|^2 + 2Ct)}{\nu \lambda_1 - K}. \]

Let \( P(t, x, A) \) be the transition probability measure of \( X \) and define

\[ \mu_T(A) = \frac{1}{T} \int_0^T P(t, x, A) \, dt. \]

For \( R > 0 \), let \( B_R = \{ x \in \mathbb{H}; \|x\| \leq R \} \), we have

\[ \mu_T(B_R^c) = \frac{1}{T} \int_0^T P(t, x, B_R^c) \, dt \leq \frac{1}{TR^2} \int_0^T \mathbb{E}\|X_t\|^2 \, dt \leq \frac{M}{R^2} \]

for some constant \( M > 0 \). Hence \( \{\mu_T, T > 0\} \) is tight and its limit is an invariant probability measure of the solution \( X \).
The Ergodicity

**Strong Feller + irreducibility**

**Lemma 1.** Suppose that Hypothesis 2. holds. This is a Markov process of (2) satisfying the Feller property on $\mathcal{D}(A^\alpha)$.

**Lemma 2.** Suppose that Hypothesis 2. holds. The solution of (2) is irreducible on $\mathcal{D}(A^\alpha)$. 
Outline of prove Lemma 1.
First prove the solution $X^R$ of the truncated equation (2) has strong Feller property, then take limitation.
we get the following Bismut-Elworthy-Li formula for stochastic 2D Navier-Stokes equation with jumps:

$$D_x \mathbb{E} \varphi(X_t(x)) \cdot h = \frac{1}{t} \mathbb{E} \left[ \varphi(X_t(x)) \int_0^t \left\langle (QQ^*)^{-1/2} \eta^h_s(x), dW_s \right\rangle \right].$$

(11)

where $\eta^h_t(x)$ is the solution of the equation:

$$d\eta^h_t(x) = - \left\{ A\eta^h_t(x) + \left[ (B(X_t(x), \eta^h_t(x)) + (B(\eta^h_t(x), X_t(x)) \right] \right\} \, dt$$
$$+ \int_{\mathbb{R}^2} Df(X_t(x), z) \cdot \eta^h_t(x) \tilde{N} (dt, dz),$$

$$\eta^h_0(\xi) = h(\xi).$$

(12)
Set

\[ \tau^k = \left\{ t > 0 : \int_0^t \| z_s \|_{L^4}^4 \, ds > k \right\}. \]

Since from the Hypothesis 2, we have For \( k \geq 1 \),

\[ \left\| (QQ^*)^{-1/2}y \right\|^2 \leq C_1 |A^{2\alpha}y|^2 \quad \forall y \in \mathcal{D}(A^{2\alpha}). \]

Therefore, we have, for any \( x, y \in \mathcal{D}(A^\alpha) \) and \( \varphi \in C_b(\mathcal{D}(A^\alpha)) \),

\[ \left| P_{t \wedge \tau^k} \varphi(x) - P_{t \wedge \tau^k} \varphi(y) \right| \leq \frac{C_1}{t} \| \varphi \|_\infty \sup_{k,h \in \mathcal{D}(A^\alpha) \atop |A^\alpha h| \leq 1} \left[ \mathbb{E} \int_0^t \left| A^{2\alpha} D_x X_s^{(R)}(k) \cdot h \right|^2 \, ds \right]^{1/2} \| x - y \|_{\mathcal{D}(A^\alpha)} \]

\[ \leq \frac{C_1(R, k, \varepsilon, \lambda_1, t)}{t} \| \varphi \|_\infty \| x - y \|_{\mathcal{D}(A^\alpha)}. \]
Since $\tau_k \uparrow \infty$, $k \to \infty$, This implies that $P_t$ is strong Feller on $\mathcal{D}(A^\alpha)$.

**Outline of prove Lemma 2.**

(i) For every $k \geq 1$, we firstly prove that the equation

$$dX^n(t) + \nu AX^n(t)dt + BX^n(t) = \int_{U_n} f(X^k(t-), u) \tilde{N}(dt, du)$$

$$+ \sqrt{Q}dW_t$$

is irreducible.

(ii) Let the transition probability of $X^n(t)$ is $P^0_t(x, C)$ and $P^f_t(x, C)$ be the above equation
\[ P_t^f(x, C) = e^{-t\lambda(U)}P_t^0(x, C) + \int_0^t \int \int H U e^{-s\lambda(U)} P_{t-s}^f(y + f(y, z), C)\lambda(dz)P_s^0(x, dy). \]

Then we can prove the solution of the second equation is irreducible.

(iii) Lastly for \( y \in \mathcal{D}(A^\alpha) \),

\[
\mathbb{P}\{\|X_t(x) - y\|_{\mathcal{D}(A^\alpha)} \geq 2\varepsilon\} \\
\leq \mathbb{P}\{\|X_t(x) - X^n_t(x)\|_{\mathcal{D}(A^\alpha)} \geq \varepsilon\} + \mathbb{P}\{\|X^n_t(x) - y\|_{\mathcal{D}(A^\alpha)} \geq \varepsilon\} \\
< 1.
\]

Then \( X_t(x) \) is irreducible.
Reference


Thank You!