Approximations for SDE's driven by Lévy processes

Arturo Kohatsu-Higa (arturokohatsu@gmail.com)[1]

July 2, 2009

[1] Osaka University. This presentation follows from two papers one written jointly with P. Tankov and another with H. Tanaka

Abstract
Approximating a SDE driven by a Lévy process involve approximating the jump part and the Brownian part. The rate of convergence of SDE driven by Brownian motion is well understood. In the first part of our talk, we discuss the rate of convergence of a pure jump Lévy process.
This will indicate what is the right effort that one has to invest in order to achieve some prescribed error. The difficult question is how to perform an analysis of such a scheme.
We use the Kusuoka scheme method developed for continuous diffusions and apply it to Lévy driven SDE's. This leads to an algebraic decomposition method which is of independent interest.

Setting

$$X_t(x) = x + \int_0^t \tilde{V}_0(X_{s-}(x))ds + \int_0^t V(X_{s-}(x))dB_s + \int_0^t h(X_{s-}(x))dY_s. \quad (1)$$

with C_b^∞ coefficients
$\tilde{V}_0 : \mathbf{R}^N \to \mathbf{R}^N$, $V = (V_1, \dots, V_d)$, $h : \mathbf{R}^N \to \mathbf{R}^N \otimes \mathbf{R}^d$
B_t is a d-dim. BM and Y_t is an d-dim. Lévy with triplet $(b, 0, \nu)$ satisfying the condition

$$\int_{\mathbf{R}_0^d} (1 \wedge |y|^p)\nu(dy) < \infty.$$

for any $p \in \mathbb{N}$.

Goal

Our purpose is to find discretization schemes $(X_t^{(n)}(x))_{t=0,T/n,\dots,T}$ for given $T > 0$ such that

$$|E[f(X_T(x))] - E[f(X_T^{(n)}(x))]| \le \frac{C(T, f, x)}{n^m}.$$

0a. Known general methods: Approximate by compounded Poisson (ignore small jumps). Protter, Talay, Kurtz, Meleard, Mordecki, Spessezy, et al.
Adaptive Weak Approximation of Diffusions with Jumps by E. Mordecki, A. Szepessy, R. Tempone and G. E. Zouraris
Let

$$Y_t = \sum_{i=1}^{N_t} Y_i$$

Algorithm:
1. Simulate the jump times τ_i and the corresponding independent jump sizes Y_i.
2. Given a fixed deterministic partition t_i simulate the Brownian increments
3. Adaptive step: Find an expansion of the error in order to choose the fixed partition t_i "optimally".

Remarks

- Questions (We are interested in stable like Lévy measures.):
  1a. Proof for Asmussen-Rosinski type approach
  1b.Do we need to simulate all jumps if one wants an approximation of order 1 or 2? (Partial answer in the first part)
  1c.Limiting the number of jumps

Remarks

- Questions (We are interested in stable like Lévy measures.):
  1a. Proof for Asmussen-Rosinski type approach
  1b.Do we need to simulate all jumps if one wants an approximation of order 1 or 2? (Partial answer in the first part)
  1c.Limiting the number of jumps
- 2a. First, we consider the case where $\tilde{V}_0 = V_i = 0$ (joint with Peter Tankov) to study the rate of convergence due to jumps only.
  2b. Once this is done then one can incorporate the Brownian motion and/or drift so that each component lead to the same order of error. (joint with Hideyuki Tanaka)

Remarks

- Questions (We are interested in stable like Lévy measures.):
  1a. Proof for Asmussen-Rosinski type approach
  1b.Do we need to simulate all jumps if one wants an approximation of order 1 or 2? (Partial answer in the first part)
  1c.Limiting the number of jumps
- 2a. First, we consider the case where $\tilde{V}_0 = V_i = 0$ (joint with Peter Tankov) to study the rate of convergence due to jumps only.
  2b. Once this is done then one can incorporate the Brownian motion and/or drift so that each component lead to the same order of error. (joint with Hideyuki Tanaka)
- 2a. Proof through a variation the classical method
  2b. Proof through a "semigroup type approach"

Remarks

- Questions (We are interested in stable like Lévy measures.):
  1a. Proof for Asmussen-Rosinski type approach
  1b.Do we need to simulate all jumps if one wants an approximation of order 1 or 2? (Partial answer in the first part)
  1c.Limiting the number of jumps
- 2a. First, we consider the case where $\tilde{V}_0 = V_i = 0$ (joint with Peter Tankov) to study the rate of convergence due to jumps only.
  2b. Once this is done then one can incorporate the Brownian motion and/or drift so that each component lead to the same order of error. (joint with Hideyuki Tanaka)
- 2a. Proof through a variation the classical method
  2b. Proof through a "semigroup type approach"
- 3b. Ideas come from Kusuoka type approximations
  S. Kusuoka. Approximation of expectation of diffusion processes based on Lie algebra and Malliavin calculus. Advances in mathematical economics. Vol. 6, 69-83, Adv. Math. Econ., 6,

One dim. SDE's driven by pure jump Lévy process

Y is a $(b, 0, \nu)$ Lévy process.
$\lambda_\varepsilon := \int_{|z|>\varepsilon} \nu(dy)$. We suppose that $\lim_{\varepsilon \to 0} \lambda_\varepsilon = \infty$.
$b_\varepsilon := b - \int_{\varepsilon<|z|\leq 1} z\nu(dz)$. $\sigma_\varepsilon^2 := \int_{|z|\leq\varepsilon} z^2\nu(dz)$

$$Y_t = bt + \int_0^t \int_{|y|\leq 1} y\widehat{N}(dy, ds) + \int_0^t \int_{|y|>1} yN(dy, ds), \quad t \in [0, 1].$$

Let $\hat{X}$ be the scheme:
1. Consider only jumps bigger than ε. (jump times T_i^ε)
2. Approximate the small jumps with the Asmussen-Rosinski approach
3. Solve as explicitely as possible the SDE with small noise between jump times and add in the jumps

$$d\hat{X}_t = h(\hat{X}_{t-}) \left\{ dZ_t^\varepsilon + \sigma_\varepsilon dW_t + b_\varepsilon dt + \frac{1}{2}(h'(\hat{X}_t) - h'(\hat{X}_{\eta(t)}))\sigma_\varepsilon^2 dt \right\}.$$

The solution is computed with

$$dX_t = h(X_t)dt, \quad X_0 = x. \tag{2}$$

In the one-dimensional case, the solution to (2) is:

$$X_t := \theta(t; x) = F^{-1}(t + F(x)),$$

where $F'(x) = \frac{1}{h(x)}$.
We define $\hat{X}(0) = X_0$ and for $i \geq 1$, $t \in (T_i^\varepsilon, T_{i+1}^\varepsilon]$, we define

$$\hat{X}(t) = \theta\Big(b_\varepsilon(t - \eta_t) + \sigma_\varepsilon(W(t) - W(\eta_t)) - \frac{1}{2}h'(\hat{X}(\eta_t))\sigma_\varepsilon^2(t - \eta_t); \hat{X}(\eta_t)\Big), \tag{3}$$

where $\eta_t := \sup\{T_i^\varepsilon : T_i^\varepsilon \leq t\}$. Here W denotes a 1dim BM

$$\hat{X}(T_i^\varepsilon) = \hat{X}(T_i^\varepsilon -) + h(\hat{X}(T_i^\varepsilon -))\Delta Z(T_i^\varepsilon).$$

($\mathbf{H_n}$) $f \in C_b^n$, $h \in C_b^n$ and $\int |z|^{2n}\nu(dz) < \infty$.
($\mathbf{H'_n}$) $f \in C_p^n$, $h \in C_p^n$, and $\int |z|^k \nu(dz) < \infty$ for all $k \geq 1$.

Theorem

(i) *Assume* ($\mathbf{H_3}$) *or* ($\mathbf{H'_3}$). *Then*

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C\left(\frac{\sigma_\varepsilon^2}{\lambda_\varepsilon}(\sigma_\varepsilon^2 + |b_\varepsilon|) + \int_{|y|\leq\varepsilon} |y|^3 \nu(dy)\right).$$

(ii) *Assume* ($\mathbf{H_4}$) *or* ($\mathbf{H'_4}$), *let* ν *satisfy*

$$\left|\int_{|y|\leq\varepsilon} y^3 \nu(dy)\right| \leq \int_{|y|\leq\varepsilon} |y|^4 \nu_0(dy) \qquad (4)$$

for some measure ν_0 . *Then*

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C\left(\frac{\sigma_\varepsilon^2}{\lambda_\varepsilon}(\sigma_\varepsilon^2 + |b_\varepsilon|) + \int_{|y|\leq\varepsilon} |y|^4 (\nu_0 + \nu)(dy)\right).$$

Note that in this case $|b_\varepsilon|$ *is bounded.*

$$
\begin{aligned}
E[f(\hat{X}_1) - f(X_1)] &= E[u(1, \hat{X}_1) - u(0, X_0)] \\
&= \int_0^1 dt E\Bigg[\frac{\partial^2 u(t, \hat{X}_t)}{\partial x^2} \sigma_\varepsilon^2 h^2(\hat{X}_t) \\
&\quad - \int_{|x| \le \varepsilon} \lambda(dy) (u(t, \hat{X}_t + h(\hat{X}_t) y) - u(t, \hat{X}_t) - \frac{\partial u(t, \hat{X}_t)}{\partial x} h(\hat{X}_t) y) \Bigg] \\
&+ \int_0^1 dt E \left[\frac{1}{2} \sigma_\varepsilon^2 \frac{\partial u(t, \hat{X}_t)}{\partial x} h(\hat{X}_t) (h'(\hat{X}_t) - h'(\hat{X}_{\eta(t)})) \right].
\end{aligned}
$$

Example. [Stable-like behavior] Assume that

$$\nu(z) = \frac{g(z)}{|z|^{1+\alpha}},$$

where g has finite nonzero right and left limits at zero as, for example, for the tempered stable process (CGMY). Then $\sigma_\varepsilon^2 = O(\varepsilon^{2-\alpha})$, $\lambda_\varepsilon = O(\varepsilon^{-\alpha})$,

$$\int_{|y|\le\varepsilon} |y|^3 \nu(dy) = O(\varepsilon^{3-\alpha}) \quad \text{and} \quad \int_{|y|\le\varepsilon} |y|^4 \nu(dy) = O(\varepsilon^{4-\alpha}),$$

so that in general for $\alpha \in (0,2)$

$$|E[f(\hat{X}_1) - f(X_1)]| \le O(\lambda_\varepsilon^{1-\frac{3}{\alpha}}),$$

and if the Lévy measure is locally symmetric near zero and $\alpha \in (0,1)$,

$$|E[f(\hat{X}_1) - f(X_1)]| \le O(\lambda_\varepsilon^{1-\frac{4}{\alpha}}).$$

Comments and Conclusions

1. Simulating many jumps imply fast convergence rate if there is concentration of jumps around zero. But the fact that small jumps are simulated by Brownian motion are also a component of the error.

Comments and Conclusions

1. Simulating many jumps imply fast convergence rate if there is concentration of jumps around zero. But the fact that small jumps are simulated by Brownian motion are also a component of the error.
2. How to match this with the simulation of the Brownian component (Euler?)? How to carry the error analysis?
3. We also want to state other approximations on weak form.
conclusion: we need a "separating" technique (which will be called coordinate process).
Next: An alternative proof for weak approximations

Set-up for the proof of weak approximation

Define:

$$P_t f(x) = E[f(X_t(x))]$$

$Q_t \equiv Q_t^n$: operator such that the semigroup property is satisfied in $\{kT/n; k = 0, ..., n\}$.
$Q_t \approx P_t$ in the sense that $(P_t - Q_t)f(x) = \mathcal{O}(t^{m+1})$. Then the idea of the proof is

$$P_T f(x) - (Q_{T/n})^n f(x) = \sum_{k=0}^{n-1} (Q_{T/n})^k (P_{T/n} - Q_{T/n}) P_{T - \frac{k+1}{n} T} f(x).$$

For the proof to work we essentially need:
Assumption $\mathcal{R}(m, \delta_m)$: The local difference $P_{T/n} - Q_{T/n}$ has to be a "small" operator.
Assumption $(\mathcal{M})$: The operators $(Q_{T/n})^k$ and $P_{T - \frac{k+1}{n} T}$ have to be stable.
Next; We need to find a stochastic representation for Q and interpret the composition.

Simulation (stochastic characterization): Let $M = M_t(x)$ s.t. $Q_t f(x) = E[f(M_t(x))]$. Then

$$Q_T f(x) = (Q_{T/n})^n f(x) = E[f(M^1_{T/n} \circ \cdots \circ M^n_{T/n}(x))]$$

Example: Euler-Maruyama scheme:
$M_t(x) := x + \tilde{V}_0(x)t + V(x)B_t + h(x)Y_t$ satisfies
Next: One has to be able to find stochastic approximations from the form of $P_{T/n}$. The idea is to approximate the generators.

Assumption $\mathcal{R}(m, \delta_m)$: The local difference $P_t - Q_t$ has to be a "small" operator for small t.

Assumption $(\mathcal{M})$: The operators $(Q_{T/n})^k$ and $P_{T-\frac{k+1}{n}T}$ have to be stable.
From now on: think that t is small (say $t = T/n$)

The algebraic structure

$$P_t = e^{tL} = \sum_{k=0}^{m} \frac{t^k}{k!} L^k + \mathcal{O}(t^{m+1})$$

Note that $L = \sum_{i=0}^{d+1} L_i$.

$$P_t^i = e^{tL_i} = \sum_{k=0}^{m} \frac{t^k}{k!} L_i^k + \mathcal{O}(t^{m+1})$$

Goal: Approximate e^{tL}, through a combination of L_i's s.t.

$$e^{tL} - \sum_{j=1}^{k} \xi_j e^{t_{1,j} A_{1,j}} \cdots e^{t_{\ell_j,j} A_{\ell_j,j}} = \mathcal{O}(t^{m+1})$$

$t_{i,j} > 0$, $A_{i,j} \in \{L_0, L_1, \dots, L_{d+1}\}$ and weights $\{\xi_j\} \subset [0,1]$ with $\sum_{j=1}^{k} \xi_j = 1$. If needed one may need to further approximate each $e^{t_{1,j} A_{1,j}}$ (m-th order scheme).

How does the algebraic argument work?

For simplicity let $d+1=2$ then

$$\begin{aligned} e^{tL} &= I + tL + \frac{t^2}{2}L^2 + O(t^3) \\ e^{tL_1}e^{tL_2} &\approx (I + tL_1 + \frac{t^2}{2}L_1^2 + \ldots)(I + tL_2 + \frac{t^2}{2}L_2^2 + \ldots) \\ &= I + tL + \frac{t^2}{2}\left(L_2^2 + L_1^2 + L_1L_2\right) + O(t^3) \end{aligned}$$

then

$$\begin{aligned} e^{tL} - e^{tL_1}e^{tL_2} &= O(t^2) \\ e^{tL} - \frac{1}{2}e^{tL_1}e^{tL_2} - \frac{1}{2}e^{tL_2}e^{tL_1} &= O(t^3). \end{aligned}$$

Finally one needs to obtain a stochastic representation for $\frac{1}{2}e^{tL_1}e^{tL_2} + \frac{1}{2}e^{tL_2}e^{tL_1}$ and possibly approximate each "coordinate".

Examples of schemes:
Ninomiya-Victoir (a):

$$\frac{1}{2} e^{\frac{t}{2} L_0} e^{tL_1} \cdots e^{tL_{d+1}} e^{\frac{t}{2} L_0} + \frac{1}{2} e^{\frac{t}{2} L_0} e^{tL_{d+1}} \cdots e^{tL_1} e^{\frac{t}{2} L_0}$$

Ninomiya-Victoir (b):

$$\frac{1}{2} e^{tL_0} e^{tL_1} \cdots e^{tL_{d+1}} + \frac{1}{2} e^{tL_{d+1}} \cdots e^{tL_1} e^{tL_0}$$

Splitting (Strang) method:

$$e^{\frac{t}{2} L_0} \cdots e^{\frac{t}{2} L_d} e^{tL_{d+1}} e^{\frac{t}{2} L_d} \cdots e^{\frac{t}{2} L_0}$$

So the idea is

$$P_t f = e^{tL} f \approx \sum_{j=1}^{k} \xi_j e^{t_{1,j} A_{1,j}} \cdots e^{t_{\ell_j,j} A_{\ell_j,j}} f$$

$$\approx \sum_{j=1}^{k} \xi_j E\left[f\left(M_1(t_{1,j}, M_2(t_{2,j}, (\ldots., M_l(t_{l,j}, \cdot))\ldots))\right)\right]$$

First example: Coordinate processes

Define the coordinate processes $X_{i,t}(x)$, $i = 0, ..., d+1$, solutions of

$$X_{0,t}(x) = x + \int_0^t V_0(X_{0,s}(x))ds$$

$$X_{i,t}(x) = x + \int_0^t V_i(X_{i,s}(x)) \circ dB_s^i \quad 1 \le i \le d$$

$$X_{d+1,t}(x) = x + \int_0^t h(X_{d+1,s-}(x))dY_s.$$

Define

$$Q_{i,t}f(x) := E[f(X_{i,t}(x))]$$

whose generators are $(\tau(y) = y1|y| \le 1$

$$L_0 f(x) := (V_0 f)(x), \quad L_i f(x) := \frac{1}{2}(V_i^2 f)(x), \ 1 \le i \le d$$

$$L_{d+1}f(x) := \nabla f(x)h(x)b + \int (f(x + h(x)y) - f(x) - \nabla f(x)h(x)\tau(y))\nu(dy)$$

$C_p^m \equiv C_p^m(\mathbf{R}^N)$: the set of C^m functions $f : \mathbf{R}^N \to \mathbf{R}$ s.t. for each multi-index α with $0 \leq |\alpha| \leq m$, $|\partial_x^\alpha f(x)| \leq C(\alpha)(1 + |x|^p)$ for some positive constant $C(\alpha)$.

General framework

Assumptions $\mathcal{M}$

- If $f \in C_p$ with $p \geq 2$, then $Q_t f \in C_p$ and

$$\sup_{t \in [0,T]} \|Q_t f\|_{C_p} \leq K \|f\|_{C_p}$$

for $K > 0$ independent of n. Futhermore, we assume $0 \leq Q_t f(x) \leq Q_t g(x)$ whenever $0 \leq f \leq g$.

General framework

Assumptions $\mathcal{M}$

- If $f \in C_p$ with $p \geq 2$, then $Q_t f \in C_p$ and

$$\sup_{t \in [0,T]} \|Q_t f\|_{C_p} \leq K \|f\|_{C_p}$$

for $K > 0$ independent of n. Futhermore, we assume $0 \leq Q_t f(x) \leq Q_t g(x)$ whenever $0 \leq f \leq g$.

- For $f_p(x) := |x|^{2p}$ $(p \in \mathbf{N})$,

$$Q_t f_p(x) \leq (1 + Kt) f_p(x) + K' t$$

for $K = K(T,p)$, $K' = K'(T,p) > 0$.

General framework

Assumptions $\mathcal{M}$

- If $f \in C_p$ with $p \geq 2$, then $Q_t f \in C_p$ and

$$\sup_{t \in [0,T]} \|Q_t f\|_{C_p} \leq K \|f\|_{C_p}$$

  for $K > 0$ independent of n. Futhermore, we assume $0 \leq Q_t f(x) \leq Q_t g(x)$ whenever $0 \leq f \leq g$.
- For $f_p(x) := |x|^{2p}$ $(p \in \mathbf{N})$,

$$Q_t f_p(x) \leq (1 + Kt) f_p(x) + K' t$$

  for $K = K(T, p)$, $K' = K'(T, p) > 0$.
- For $m \in \mathbf{N}$, $\delta_m : [0, T] \to \mathbf{R}_+$ denotes a decreasing function s.t.

$$\limsup_{t \to 0+} \frac{\delta_m(t)}{t^{m-1}} = 0.$$

  Usually, we have $\delta_m(t) = t^m$.

Main hypothesis $\mathcal{R}(m, \delta_m)$

For $p \geq 2$, there exists $q = q(m,p) \geq p$ and linear operators $e_k : C_p^{2k} \to C_{p+2k}$ $(k = 0, 1, \ldots, m)$ s.t.

(A): For every $f \in C_p^{2(m'+1)}$ with $1 \leq m' \leq m$, the operator Q_t satisfies

$$Q_t f(x) = \sum_{k=0}^{m'} (e_k f)(x) t^k + (\mathrm{Err}_t^{(m')} f)(x), \;\; t \in [0, T], \tag{5}$$

where $\mathrm{Err}_t^{(m')} f \in C_q$, and satisfies the following condition:

(B): If $f \in C_p^{m''}$ with $m'' \geq 2k$, then $e_k f \in C_{p+2k}^{m''-2k}$ and there exists a constant constant $K = K(T, m) > 0$ such that

$$\|e_k f\|_{C_{p+2k}^{m''-2k}} \leq K \|f\|_{C_p^{m''}} \;\; k = 0, 1, \ldots, m. \tag{6}$$

Furthermore if $f \in C_p^{m''}$ with $m'' \geq 2m' + 2$,

$$\|\mathrm{Err}_t^{(m')} f\|_{C_q} \leq \begin{cases} K t^{m'+1} \|f\|_{C_p^{m''}} & \text{if } m' < m \\ K t \delta_m(t) \|f\|_{C_p^{m''}} & \text{if } m' = m \end{cases}$$

for all $0 \leq t \leq T$.

Weak approximation result

(C): For every $0 \leq k \leq m$ and $j \geq 2k+2$, if $f \in C_p^{1,j}([0,T] \times \mathbf{R}^N)$, then $e_k f \in C_{p+2k}^{1,j-2k}([0,T] \times \mathbf{R}^N)$.

Define $J_{\leq m}(Q_t)(f)(x) = \sum_{k=0}^{m} (e_k f)(x) t^k$

Theorem

Assume $(\mathcal{M})$ and $\mathcal{R}(m, \delta_m)$ for P_t and Q_t with $J_{\leq m}(P_t - Q_t) = 0$. Then for any $f \in C_p^{2(m+1)}$, there exists a constant $K = K(T, x) > 0$ such that

$$\left| P_T f(x) - (Q_{T/n})^n f(x) \right| \leq K \delta_m \left(\frac{T}{n} \right) \|f\|_{C_p^{2(m+1)}}. \tag{7}$$

Theorem

Assume $(\mathcal{M})$ and $\mathcal{R}(m+1, \delta_{m+1})$ for Q_t with $J_{\leq m}(P_t - Q_t) = 0$. Then for each $f \in C_p^{2(m+3)}$, we have

$$P_T f(x) - (Q_{T/n})^n f(x) = \frac{K}{n^m} + \mathcal{O}\left(\left(\frac{T}{n} \right)^{m+1} \vee \delta_{m+1} \left(\frac{T}{n} \right) \right) \tag{8}$$

Combination of "coordinates" and their approximation

Theorem

Assume $(\mathcal{M})$ and $\mathcal{R}(2,\delta_2)$ are satisfied for $Q_t^{\bar{X}_i}$ $(i=0,1,\dots,d+1)$ associated with indep. processes $\bar{X}_0,\dots,\bar{X}_{d+1}$ with $J_{\le 2}(Q_{i,t}-Q_t^{\bar{X}_i})=0$. Then all the following operators satisfy $(\mathcal{M})$ and $\mathcal{R}(2,\delta_2)$:

N-V(a) $$Q_t^{(a)}=\frac{1}{2}Q_{t/2}^{\bar{X}_0}\prod_{i=1}^{d+1}Q_t^{\bar{X}_i}Q_{t/2}^{\bar{X}_0}+\frac{1}{2}Q_{t/2}^{\bar{X}_0}\prod_{i=1}^{d+1}Q_t^{\bar{X}_{d+2-i}}Q_{t/2}^{\bar{X}_0}$$

N-V(b) $$Q_t^{(b)}=\frac{1}{2}\prod_{i=0}^{d+1}Q_t^{\bar{X}_i}+\frac{1}{2}\prod_{i=0}^{d+1}Q_t^{\bar{X}_{d+1-i}}$$

Splitting $$Q_t^{(sp)}=Q_{t/2}^{\bar{X}_0}\cdots Q_{t/2}^{\bar{X}_d}Q_t^{\bar{X}_{d+1}}Q_{t/2}^{\bar{X}'_d}\cdots Q_{t/2}^{\bar{X}'_0}$$

where $(\bar{X}'_0,\dots,\bar{X}'_d)$ is a further indep. copy of $(\bar{X}_0,\dots,\bar{X}_d)$.
Moreover, we have
$J_{\le 2}(Q_t^{(a)})=J_{\le 2}(Q_t^{(b)})=J_{\le 2}(Q_t^{(sp)})=\sum_{k=0}^{2}\frac{t^k}{k!}L^k$. In particular, the above schemes define a second order approximation scheme.

Recall the set-up:
1. $P_t = e^{tL} = \exp\left(\sum_{i=1}^{n} tL_i\right)$.
2. Through an algebraic combination is enough to obtain an approximation for $\exp(tL_i)$ for small t. In fact, one may even consider $\exp(tL_i^t)$ and try to use the same algebraic combination.
3. Given an order of error t^M (think $M = 3$ for simplicity). First, obtain the algebraic combination that will give you this error. Then approximate each term in the algebraic decomposition up to this error.
4. Put the whole approximation scheme together.

First Example (Diffusion coordinate)

Theorem

Let $t > 0$*.* $A^i = B^i_t V_i : \mathbf{R}^N \to \mathbf{R}^N \in C^\infty_b$*. Let* z *be the exponential map defined as the solution of the ode*

$$\frac{dz^i_s(x)}{dt} = A^i(z_s(x)),\ z^i_0(x) = x,\ s \in [0,1]. \tag{9}$$

For $i = 0, 1, ..., d$*, the sde*

$$X_{i,t}(x) = x + \int_0^t V_i(X_{i,s}(x)) \circ dB^i_s \tag{10}$$

has a unique solution given by

$$X_{i,t}(x) = z^i_1(x).$$

Example: Compound Poisson

$$Y_t = \sum_{i=1}^{N_t} J_i$$

where (N_t) : Poisson (λ) and (J_i) are i.i.d. $\mathbf{R}^d$-r.v. indep. of (N_t) with $J_i \in \bigcap_{p\geq 1} L^p$.
In this case Y_t is a Lévy process with generator of the form

$$L^{d+1}f(x) = \int_{\mathbf{R}_0^d} (f(x+y) - f(x))\nu(dy)$$

where $\tau \equiv 0$, $b = 0$, $\nu(\mathbf{R}_0^d) = \lambda < \infty$ and $\nu(dy) = \lambda P(J_1 \in dy)$.
Then in this case

$$X_t^{d+1}(x) = x + \int_0^t h(X_{s-}^{d+1}(x))dY_s, \ t \in [0, T] \qquad (11)$$

which can be solved explicitly.

Indeed, let $(G_i(x))$ be defined by recursively

$$G_0 = x$$

$$G_i = G_{i-1} + h(G_{i-1})J_i.$$

Then the solution can be written as $X_t^{d+1}(x) = G_{N_t}(x)$. Define for fixed $M \in \mathbf{N}$, the approximation process $\bar{X}_{d+1,t} = G_{N_t \wedge M}(x)$. This approximation requires the simulation of at most M jumps. In fact, the rate of convergence is fast as the following result shows. **Proposition** Let $M \in \mathbf{N}$. Then the process $G_{N_t \wedge M}(x)$ satisfies $(\mathcal{M})$ and $\mathcal{R}(M, t^{M-\kappa})$ for arbitrary small $\kappa > 0$. Furthermore $J_{\leq M}(Q_{d+1,t} - Q_t^{\bar{X}_{d+1}}) = 0$.

Infinite activity approximated by a process with no small jumps

Define for $\varepsilon > 0$ Lévy proc. (Y_t^ε) with Lévy triplet $(b, 0, \nu^\varepsilon)$

$$\nu^\varepsilon(E) := \nu(E \cap \{y : |y| > \varepsilon\}), \; E \in \mathcal{B}(\mathbf{R}_0^d). \tag{12}$$

Consider the approximate coordinate SDE

$$\bar{X}_{d+1,t}(x) = x + \int_0^t h(\bar{X}_{d+1,s-}(x)) dY_s^\varepsilon,$$

$$L_{d+1}^{1,\varepsilon} f(x) = \nabla f(x) h(x) b + \int (f(x+h(x)y) - f(x) - \nabla f(x) h(x) \tau(y)) \nu^\varepsilon(dy)$$

Now we derive the error estimate for $\bar{X}_{d+1,t}$.

Theorem

Assume that $\sigma^2(\varepsilon) := \int_{|y| \le \varepsilon} |y|^2 \nu(dy) \le t^{M+1}$ for $\varepsilon \equiv \varepsilon(t) \in (0,1]$. Then we have that $Q_t^{\bar{X}_{d+1}}$ satisfies $(\mathcal{M})$ and $\mathcal{R}(M, t^M)$. Furthermore $J_{\le M}(Q_{d+1,t} - Q_t^{\bar{X}_{d+1}}) = 0$.

Asmussen-Rosinski type approximation

Consider the new approximate SDE

$$\bar{X}_{d+1,t}(x) = x + \int_0^t h(\bar{X}_{d+1,s}(x))\Sigma_\varepsilon^{1/2} dW_s + \int_0^t h(\bar{X}_{d+1,s-}(x)) dY_s^\varepsilon$$

where W_t is a new d-dim.BM indep. of B_t and Y_t^ε, and $\Sigma_\varepsilon = \int_{|y|\le\varepsilon} yy^* \nu(dy)$.

Theorem

Assume that $0 < \varepsilon \equiv \varepsilon(t) \le 1$ *is chosen as to satisfy that* $\int_{|y|\le\varepsilon} |y|^3 \nu(dy) \le t^{M+1}$*. Then we have that* $Q_t^{\bar{X}_{d+1}}$ *satisfies* $(\mathcal{M})$ *and* $\mathcal{R}(M, t^M)$*. Furthermore* $J_{\le M}(Q_{d+1,t} - Q_t^{\bar{X}_{d+1}}) = 0$*.*

Example: Other decompositions with at most one jump per interval

$\tau(y) = y1_{|y|<1}$, assume that $\int_{|y|<1} |y|\nu(dy) < \infty$.
Then we decompose the operator

$$L_{d+1} = L^1_{d+1} + L^2_{d+1} + L^3_{d+1}$$

$$L^1_{d+1}f(x) := \nabla f(x)h(x)\left(b - \int_{\varepsilon<|y|\leq 1} y\nu(dy)\right)$$

$$L^2_{d+1}f(x) := \int_{|y|\leq\varepsilon}(f(x+h(x)y) - f(x) - \nabla f(x)h(x)y)\nu(dy)$$

$$L^3_{d+1}f(x) := \int_{\varepsilon<|y|}\{f(x+h(x)y) - f(x)\}\,\nu(dy).$$

The operator L^1_{d+1} can be exactly generated using
$\bar{X}^1_{d+1,t} = x + \left(b - \int_{\varepsilon<|y|\leq 1} y\nu(dy)\right)\int_0^t h\left(\bar{X}^1_{d+1,s}\right)ds$. Therefore we only need to approximate L^2_{d+1} and L^3_{d+1}.

Approximation for L^2_{d+1}.

$$L^2_{d+1} f(x) := \int_{|y|\leq\varepsilon} (f(x+h(x)y) - f(x) - \nabla f(x) h(x) y)\nu(dy)$$

$$\bar{X}^{2,\varepsilon}_t(x) = x + h(x) W_t \sqrt{\lambda_\varepsilon}$$

W is a d-dim. BM with cov. matrix given by $\Sigma_{ij} = |Y^\varepsilon|^{-r} Y^\varepsilon_i Y^\varepsilon_j$ which is indep. of everything else.

$$Y_\varepsilon \sim F_\varepsilon$$

$$F_\varepsilon(dy) = \lambda_\varepsilon^{-1} |y|^r 1_{|y|\leq\varepsilon} \nu(dy)$$

$$\lambda_\varepsilon = \int_{|y|\leq\varepsilon} |y|^r \nu(dy) < \infty$$

Lemma

(*)1.Assume that $\int_{|y|\le\varepsilon}|y|^3\nu(dy)\le Ct$ *and* $\sup_{\varepsilon\in(0,1]}\int_{|y|\le\varepsilon}|y|^{4-r}\nu(dy)<\infty$ *then*

$$\left|E\left[f(\bar{X}_t^{2,\varepsilon})\right]-f(x)-tL_{d+1}^2 f(x)\right|\le\|f\|_{C_p^2}(1+|x|^{p+2})t^2.$$

That is, condition $\mathcal{R}(2,t^2)$ *is satisfied.*

2. Assume that $\sup_{\varepsilon\in(0,1]}\int_{|y|\le\varepsilon}|y|^{2+\frac{(2-r)(p-2)}{2}}\nu(dy)<\infty$*, then assumption* $(\mathcal{M})$ *is satisfied with*

$$E\left[\left|\bar{X}_{d+1}^{2,\varepsilon}(x)\right|^p\right]\le(1+Kt)|x|^p+K't$$

for all $p\ge 2$*.*

The approximation for L_{d+1}^3

$$L_{d+1}^3 f(x) := \int_{\varepsilon<|y|} \{f(x+h(x)y) - f(x)\}\, \nu(dy)$$

$$\bar{X}_t^{3,\varepsilon}(x) = \begin{cases} x & \text{if } S^\varepsilon = 0 \\ x + h(x) Z^\varepsilon & \text{if } S^\varepsilon = 1 \end{cases}$$

$$Z^\varepsilon \sim G_\varepsilon$$

$$G_\varepsilon(dy) = C_\varepsilon^{-1} 1_{|y|>\varepsilon} \nu(dy)$$

$$C_\varepsilon = \int_{|y|>\varepsilon} \nu(dy)$$

S^ε is a Bernoulli r.v. indep. of Z^ε with

$$\left| C_\varepsilon^{-1} P\left[S^\varepsilon = 1\right] - t \right| \leq C t^2$$

Lemma

(**)1. Assume that $\left| C_\varepsilon^{-1} P\left[S^\varepsilon = 1\right] - t \right| \leq Ct^2$ *then*

$$\left| E\left[f(\bar{X}_t^{3,\varepsilon}) \right] - f(x) - tL_{d+1}^3 f(x) \right| \leq Ct^2 \left\| f \right\|_{C_p^1} (1+|x|^{p+1}) \int_{|y|>\varepsilon} |y| \nu(dy)$$

That is, condition $\mathcal{R}(2, t^2)$ *is satisfied.*

2. If $C_\varepsilon^{-1} P\left[S^\varepsilon = 1\right] \leq Ct$ *then assumption* $(\mathcal{M})$ *is satisfied with*

$$E\left[\left| \bar{X}_{d+1}^{3,\varepsilon}(x) \right|^p \right] \leq (1 + Kt)|x|^p + K't$$

for all $p \geq 2$.

Approximation for L^2_{d+1}: Weighted version $l : \mathbf{R}^d \to \mathbf{R}_+$ (Importance sampling)

$$L^2_{d+1} f(x) := \int_{|y|\leq\varepsilon} (f(x+h(x)y) - f(x) - \nabla f(x) h(x) y)\nu(dy)$$

$$\bar{X}^{2,\varepsilon}_t(x) = x + h(x) W_t \sqrt{\lambda_\varepsilon}$$

$$Y_\varepsilon \sim F_\varepsilon$$

$$F^l_\varepsilon(dy) = \lambda_\varepsilon l(y) 1_{|y|\leq\varepsilon} \nu(dy)$$

$$\lambda_\varepsilon^{-1} = \int_{|y|\leq\varepsilon} l(y)\nu(dy)$$

W is a d-dim. BM with cov. matrix given by $\Sigma_{ij} = l(Y^\varepsilon)^{-1} Y^\varepsilon_i Y^\varepsilon_j$ which is indep. of everything else.

Lemma

1. Assume that $\int_{|y|\leq\varepsilon}|y|^3\nu(dy)\leq Ct$ *and* $\sup_{\varepsilon\in(0,1]}\int_{|y|\leq\varepsilon}|y|^4 l(y)^{-1}\nu(dy)<\infty$ *then*

$$\left|E\left[f(\bar{X}_t^{2,\varepsilon})\right]-f(x)-tL_{d+1}^2 f(x)\right|\leq C\left\|f\right\|_{C_p^2}(1+|x|^{p+2})t^2.$$

That is, condition $\mathcal{R}(2,t^2)$ *is satisfied.*
2.Assume that $\sup_{\varepsilon\in(0,1]}\int_{|y|\leq\varepsilon}|y|^p l(y)^{-\frac{p-2}{2}}\nu(dy)<\infty$, *then assumption* $(\mathcal{M})$ *is satisfied with*

$$E\left[\left|\bar{X}_{d+1}^{2,\varepsilon}(x)\right|^p\right]\leq(1+Kt)|x|^p+K't$$

for all $p\geq 2$.

The approximation for L^3_{d+1}: Localized version

$$L^3_{d+1} f(x) := \int_{\varepsilon < |y|} \{ f(x + h(x)y) - f(x) \} \, \nu(dy)$$

$$\bar{X}^{3,\varepsilon}_t(x) = \begin{cases} x & \text{if } S^\varepsilon = 0 \\ x + h(x) l(Z^{\varepsilon,l})^{-1} Z^{\varepsilon,l} & \text{if } S^\varepsilon = 1 \end{cases}$$

$$Z^{\varepsilon,l} \sim G_{\varepsilon,l}$$

$$G_{\varepsilon,l}(dy) = C^{-1}_{\varepsilon,l} l(y) 1_{|y|>\varepsilon} \nu(dy)$$

$$C_{\varepsilon,l} = \int_{|y|>\varepsilon} l(y) \nu(dy)$$

S^ε is a Bernoulli r.v. indep. of $Z^{\varepsilon,l}$ with

$$\left| C^{-1}_{\varepsilon,l} P\left[S^{\varepsilon,l} = 1 \right] - t \right| \leq Ct^2$$

Lemma

1.Assume that
$\int_{|y|>\varepsilon} |y|^2(l(y)^{-1}-1)+|y|^{p+3}|l(y)^{-1}-1|^{p+2}\nu(dy) \leq Ct$ *and*
$\left|C_{\varepsilon,l}^{-1}P\left[S^{\varepsilon,l}=1\right]-t\right| \leq Ct^2$ *then*

$$\left|E\left[f(\bar{X}_t^{3,\varepsilon,l})\right]-f(x)-tL_{d+1}^3 f(x)\right| \leq Ct^2 \|f\|_{C_p^2}(1+|x|^{p+2}).$$

That is, condition $\mathcal{R}(2,t^2)$ *is satisfied.*
2. Assume that
$\sup_{\varepsilon\in(0,1]}\max_{j=1,\ldots,p}\int_{|y|>\varepsilon} l(y)^{1-j}|y|^j\,\nu(dy)<\infty$*.then assumption* $(\mathcal{M})$ *is satisfied with*

$$E\left[\left|\bar{X}_{d+1}^{3,\varepsilon}(x)\right|^p\right] \leq (1+Kt)|x|^p+K't$$

for all $p\geq 2$*.*

Example: Tempered stable

Let a Lévy measure ν defined on $\mathbf{R}_0$ be given by

$$\nu(dy) = \frac{1}{|y|^{1+\alpha}} \left(c_+ e^{-\lambda_+ |y|} 1_{y>0} + c_- e^{-\lambda_- |y|} 1_{y<0} \right) dy$$

- Gamma: $\lambda_+, c_+ > 0$, $c_- = 0$, $\alpha = 0$.
- Variance gamma: $\lambda_+, \lambda_-, c_+, c_- > 0$, $\alpha = 0$.
- Tempered stable: $\lambda_+, \lambda_-, c_+, c_- > 0$, $0 < \alpha < 2$.

Then, we have that for $\alpha \in [0, 1)$

$$\int_{|y| \le \varepsilon} |y|^k \nu(dy) \sim \varepsilon^{k-\alpha}, \quad k \ge 1.$$

Therefore $\sup_{\varepsilon \in (0,1]} \int_{|y| \le \varepsilon} |y| \nu(dy) < \infty$, then the conditions of the approximation Lemma (*) are satisfied if $r \ge \alpha$, $r + \alpha \le 4$ and $\varepsilon = t^{\frac{1}{3-\alpha}}$. approximation Lemma (**) is satisfied for example in the following case. Let $P\left[S^\varepsilon = 1\right] = e^{-C_\varepsilon a(\varepsilon, t)}$ where $a(\varepsilon, t) = -\varepsilon^\alpha \log\left(\left(t^2 + t\right) \varepsilon^{-\alpha}\right)$ as $\varepsilon = t^{\frac{1}{3-\alpha}}$ then we have that

$$a(\varepsilon(t), t) = -t^{\frac{\alpha}{3-\alpha}} \log\left((t+1) t^{\frac{3-2\alpha}{3-\alpha}}\right).$$

Comments:

- One can use importance sampling to change the jump distribution to provide an easier simulation scheme.

Comments:

- One can use importance sampling to change the jump distribution to provide an easier simulation scheme.
- Design other approximation schemes

Comments:

- One can use importance sampling to change the jump distribution to provide an easier simulation scheme.
- Design other approximation schemes
- One can also do this for $h(x, y)$

Comments:

- One can use importance sampling to change the jump distribution to provide an easier simulation scheme.
- Design other approximation schemes
- One can also do this for $h(x, y)$
- Irregular coefficients. A CIR type example. Alfonsi (2008)

Comments:

- One can use importance sampling to change the jump distribution to provide an easier simulation scheme.
- Design other approximation schemes
- One can also do this for $h(x, y)$
- Irregular coefficients. A CIR type example. Alfonsi (2008)
- Irregular functions f: Consider the right stochastic representation and concatenate. But there is a technical problem with jump type processes !

Comments:

- One can use importance sampling to change the jump distribution to provide an easier simulation scheme.
- Design other approximation schemes
- One can also do this for $h(x, y)$
- Irregular coefficients. A CIR type example. Alfonsi (2008)
- Irregular functions f: Consider the right stochastic representation and concatenate. But there is a technical problem with jump type processes !
- Relation with the Cubature method

A. Alfonsi : High order discretization schemes for the CIR process: application to Affine Term Structure and Heston models, Juin 2008.

J. Jacod, T.G. Kurtz, S. Méléard, P. Protter, *The approximate Euler method for Lévy driven stochastic differential equations*. Ann. Inst. H. Poincare Probab. Statist. 41 (2005), no. 3, 523–558.

P. Protter, D. Talay. *The Euler scheme for L?vy driven stochastic differential equations*, Ann. Proba 25, (1997) 393-423.

S.Kusuoka, M. Ninomiya, S. Ninomiya. A new weak approximation scheme of stochastic differential equations by using the Runge-Kutta method. Preprint, 2007.

T. Lyons, N. Victoir, *Cubature on Wiener space*, Proc. R. Soc. Lond. A 460, (2004) 169-198.

S. Ninomiya, N. Victoir, *Weak approximation of stochastic differential equations and application to derivative pricing*, preprint, 2005.