Riesz transforms on complete Riemannian manifolds

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Hilbert transform

Definition

Let $f \in C_0^\infty(\mathbb{R})$. The Hilbert transform of f, denoted by $\mathrm{H}f$, is defined by the following Cauchy singular integral :

$$\mathrm{H}f(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y} dy, \quad a.s.\ x \in \mathbb{R}.$$

In Fourier analysis, we can prove that, for all $f \in L^2(\mathbb{R})$,

$$\widehat{Hf}(\xi) = i\frac{\xi}{|\xi|}\widehat{f}(\xi), \quad \forall \xi \in \mathbb{R}.$$

Using the Plancherel formula, we then obtain

$$\|Hf\|_2 = \|f\|_2, \quad \forall f \in L^2(\mathbb{R}).$$

Thus, the Hilbert transform H is an isometry in L^2.

The L^p continuity of H

In 1927, Marcel Riesz proved the following remarkable

Theorem (M. Riesz 1927)

For all $p > 1$, there exists a constant $C_p > 0$ such that

$$\|\mathrm{H}f\|_p \leq C_p \|f\|_p, \quad \forall f \in L^p(\mathbb{R}).$$

According to D. L. Burkholder and P.-A. Meyer, the L^p continuity of the Hilbert transform is one of the greatest discoveries in analysis of the twenty century.

Riesz transforms on $\mathbb{R}^n$

In 1952, Caldéron and Zygmund developed the real method in the study of singular integral operators on Euclidean spaces. One of the most important results of the Calderon-Zygmund theory is the following

Theorem

For all $p > 1$, there exists a constant $C_p > 0$ independent of n such that

$$\|R_j f\|_p \leq C_p \|f\|_p, \quad \forall f \in L^p(\mathbb{R}^n),$$

where R_j, $j = 1, \ldots, n$, are the Riesz transforms on $\mathbb{R}^n$, formally defined by

$$R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}, \quad j = 1, \ldots, n.$$

Riesz transform on Gaussian spaces

Let $L = \Delta - x \cdot \nabla$ be the Ornstein-Uhlenbeck operator on the Gaussian space $(\mathbb{R}^n, d\gamma_n)$, where

$$d\gamma_n(x) = \frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{n/2}} dx.$$

Following P.-A. Meyer, the Riesz transform associated to the Ornstein-Uhlenbeck operator is defined by

$$\nabla(-L)^{-1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{tL} f \frac{dt}{\sqrt{t}},$$

where $f \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\gamma_n(f) = \int_{\mathbb{R}\ltimes} f(x) d\gamma_n(x) = 0.$$

P.-A. Meyer's inequality

Using a probabilistic approach to the Littlewood-Paley-Stein inequalities, P.-A. Meyer proved the following remarkable

Theorem (P.-A. Meyer 1982)

For all $p > 1$, there exists a constant C_p which is independent of $n = \dim \mathbb{R}^n$ such that

$$\|\nabla(-L)^{-1/2} f\|_p \leq C_p \|f\|_p, \quad \forall\, f \in L^p(\mathbb{R}^n, \gamma_n).$$

Consequently, it holds that

$$C_p^{-1}(\|f\|_p + \|\nabla f\|_p) \leq \|(1-L)^{1/2} f\|_p \leq C_p(\|f\|_p + \|\nabla f\|_p).$$

Moreover, the above inequalities remain true on the infinite dimensional Wiener space equipped with the Wiener measure.

In 1986, G. Pisier gave an analytic proof of P.-A. Meyer's inequality without using the Littlwood-Paley-Stein inequalities.

Using the P.-A. Meyer's inequalities, Airault-Malliavin and Sugita proved the following beautiful result which consists of the base of the quasi-sure analysis on infinite dimensional Wiener space (Malliavin 82/97, Ren 90, Bouleau-Hirsch 91, etc.).

Theorem (Airault-Malliavin 1990, H. Sugita 1990)

All the positive distributions (in the sense of Watanabe) on the Wiener space are probability measures.

Riesz transforms on Riemannian manifolds

Let (M, g) be a complete Riemannian manifold, ∇ the gradient, and Δ the Laplace-Beltrami operator on (M, g).

The Riesz transform on (M, g) is formally defined by

$$R = \nabla(-\Delta)^{-1/2}.$$

More precisely,

$$Rf = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{t\Delta}(f - E_0 f) \frac{dt}{\sqrt{t}},$$

where $E_0 f$ denotes the harmonic projection of f onto $\mathrm{Ker}(\Delta) \cap L^2(M)$.

By the Gaffney L^2 IBP, Strichartz proved that

Proposition (Strichartz JFA1983)

For all $f \in L^2(M)$, it holds

$$\|\nabla f\|_2^2 = -\langle\langle \Delta f, f\rangle\rangle = \|(-\Delta)^{1/2} f\|_2^2.$$

Therefore, for all $f \in L^2(M)$ with $E_0 f = 0$, we have

$$\|\nabla(-\Delta)^{-1/2} f\|_2 = \|f\|_2,$$

where $E_0 f$ denotes the harmonic projection of $f \in L^2(M)$ onto $\mathrm{Ker}(\Delta) \cap L^2(M)$.

Equivalently, on any complete Riemannian manifold (M, g), the Riesz transform $R = \nabla(-\Delta)^{-1/2}$ is an isometry in L^2 :

$$\|Rf\|_2 = \|f\|_2.$$

Fundamental problem in harmonic analysis

Problem (E.-M. Stein 1970, Strichartz 1983)

Under which condition on a complete non-compact Riemannian manifold, the Riesz transform is bounded in L^p for all (or some) $p > 1$ (but $p \neq 2$), that is, there exists a constant $C_p > 0$ such that

$$\|Rf\|_p \leq C_p \|f|_p, \quad \forall f \in L^p(M)?$$

Some results in the literature

- Non-compact Rienannian symmetric space of rank 1 Strichartz (JFA1983)
- Riemannian manifolds with bounded geometry condition and strictly positive Laplacian in L^2

  Lohoué (JFA85, CRAS85, CRAS90, Orsay94, MathNachr2006)
- Riemannian manifolds with Ricci curvature non-négative or bounded from below

  Bakry (87, 89), C.-J. Chen (88), Chen-Luo (88), J.-Y. Li (91), Shigekawa-Yoshida (92), Yoshida (92, 94)
- Riemaniann manifolds with doubling volume property and an on-diagonal heat kernel estimate

  Coulhon-Duong (TAMS99, CRAS01, CPAM03), Auscher-Couhon-Duong-Hofmann (ASENS04), Auscher-Coulhon (ASNSP05), Carron-Coulhon-Hasell (DMJ06).

- Counter-examples: Lohoué(94), Coulhon-Ledoux (94), Coulhon-Duong (99), H.-Q. Li (99), Carron-Coulhon-Hassell (DMJ2006).

Littlewood-Paley-Stein functions

- The vertical Littlewood-Paley-Stein function acting on scalar functions is defined by

$$g_2(f)(x) = \left(\int_0^\infty t \left| \nabla e^{-t\sqrt{a+\Delta}} f(x) \right|^2 dt \right)^{1/2},$$

  where $f \in C_0^\infty(M)$, Δ is the Laplace-Beltrami operator acting on scalar functions.

- The horizontal Littlewood-Paley-Stein function acting on one-forms is defined by

$$g_1(\omega)(x) = \left(\int_0^\infty t \left| \frac{\partial}{\partial t} e^{-t\sqrt{a-\Box}} \omega(x) \right| dt \right)^{1/2}.$$

  where $\omega \in C_0^\infty(M, \Lambda T^* M)$, and $\Box$ denotes the Hodge Laplacian acting on one-forms.

By spectral decomposition and the Littlewood-Paley identity, it has been well-known (e.g. Chen 1985, Bakry 1987, ...) that

$$\langle\langle\nabla(a-\Delta)^{-1/2}f,\omega\rangle\rangle\leq 4\|g_2(f)\|_p\|g_1(\omega)\|_q,$$

where $\frac{1}{p}+\frac{1}{q}=1$. Thus, if one can prove that the following two Littlewood-Paley inequalities :

$$\|g_2(f)\|_p\leq A_p\|f\|_p,\quad \|g_1(\omega)\|_q\leq B_q\|\omega\|_q,$$

then we can derive the continuity of the Riesz transform $\nabla(a-L)^{-1/2}$ on L^p

$$\|\nabla(a-\Delta)^{-1/2}f\|_p\leq 4A_pB_q\|f\|_p.$$

Bakry's work on Riesz transforms

Using a probabilistic approach to the Littlewood-Paley-Stein inequalities, Bakry proved the following remarkable

Theorem (Bakry 1987)

Let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $a \geq 0$ such that

$$Ric + \mathrm{Hess}\phi \geq -a.$$

Then, for all $\forall p \in]1, \infty[$, there exists a constant $C_p > 0$ such that

$$\|\nabla (a - L)^{-1/2} f\|_p \leq C_p \|f\|_p, \quad \forall f \in C_0^\infty(M),$$

where

$$\|f\|_p^p := \int_M |f(x)|^p e^{-\phi} dv.$$

Ricci curvature

By the Gauss lemma, in the geodesic normal coordinates near any point p in a Riemannian manifold (M, g), we have

$$g_{ij} = \delta_{ij} + O(|x|^2).$$

In these coordinates, the metric volume form then has the following Taylor expansion at p:

$$d\nu_g = \left(1 - \frac{1}{6} R_{ij} x_i x_j + O(|x|^2)\right) d\nu_{\mathrm{Euclidean}}.$$

Manifolds with unbounded Ricci curvature

Theorem (Li RMI2006)

Let (M, g) *be a complete Riemannian manifold, and* $\phi \in C^2(M)$*. Suppose that there exists a constant* $m > 2$ *such that the Sobolev inequality holds*

$$\|f\|_{\frac{2m}{m-2}} \leq C_m(\|\nabla f\|_2 + \|f\|_2), \quad \forall f \in C_0^\infty(M).$$

Suppose that there exist some constants $c \geq 0$ *and* $\epsilon > 0$ *such that*

$$(K + c)^- \in L^{\frac{m}{2}+\epsilon}(M, \mu),$$

where

$$K(x) = \inf\{\langle(Ric + \mathrm{Hess}\phi)v, v\rangle : \ v \in T_xM, \ |v| = 1\}.$$

Then, the Riesz transform $\nabla(a - L)^{-1/2}$ *is bounded in* $L^p(M, e^{-\phi}dv)$ *for all* $p \geq 2$ *and for all* $a > 0$*.*

Manifolds with unbounded Ricci curvature

Theorem (Li RMI2006)

Let (M, g) be an n-dimensional Cartan-Hadamard manifold. Suppose $(C1)$ there exist a constant $C > 0$ and a fixed point $o \in M$ such that

$$Ric(x) \geq -C[1 + d^2(o, x)], \quad \forall x \in M,$$

$(C2)$ there exist some constants $c \geq 0$ and $\epsilon > 0$ such that

$$(K + c)^- \in L^{\frac{n}{2}+\epsilon}(M),$$

where

$$K(x) = \inf\{\langle Ric(x)v, v\rangle : v \in T_xM, \ |v| = 1\}, \quad \forall x \in M.$$

Then, for all $p \geq 2$, the Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded in L^p.

Probabilistic representation formula of Riesz transforms

Let X_t be the Brownian motion on $\mathbb{R}^n$ with initial distribution dx, B_t the Brownian motion on $\mathbb{R}$ starting from $y > 0$ and with generator $\frac{d^2}{dy^2}$. Let

$$\tau_y = \inf\{t > 0 : B_t = 0\}.$$

For any $f \in C_0^\infty(\mathbb{R}^n)$, the Poisson integral of f is given by

$$u(x, y) = e^{-y\sqrt{-\Delta}} f(x) = E_{(x,y)}[f(X_{\tau_y})].$$

Duality between BM issue from ∞ and Bessel 3

In his paper Le dual de $H^1(\mathbb{R}^n)$ (LNM581, 1977), P.-A. Meyer described the duality between the Brownian motion issue from infinity and Bessel 3 as follows :

D'une manière intuitive, on peut donc dire que le retourné du processus de Bessel issu de λ_0 est le "mouvement brownien venant de l'infini et tué en 0", où

$$\lambda_0 = dx \otimes \delta_0.$$

In 1979, Gundy and Varopoulos (CRAS 289) proved that for all $f \in C_0^\infty(\mathbb{R}^n)$, one has

$$\frac{1}{2} R_j f(x) = \lim_{y \to \infty} E_y \left[\int_0^{\tau_y} \frac{\partial}{\partial x_j} u(X_s, B_s) dB_s \middle| X_{\tau_y} = x \right].$$

In 1982, Gundy and Silverstein reproved this formula using the time-reversal technique.

Representation of Riesz transforms on Riemannian manifolds

- (M, g) a complete Riemannian manifold, $\phi \in C^2(M)$,

$$L = \Delta - \nabla\phi \cdot \nabla, \quad d\mu = e^{-\phi} dv,$$

- the Bakry-Emery Ricci curvature associated with L is defined by

$$Ric(L) := Ric + \nabla^2\phi,$$

- X_t the L-diffusion on M with initial distribution μ,
- B_t the Brownian motion on $\mathbb{R}$ issue from $y > 0$ and with $E[B_t^2] = 2t$. Set

$$\tau_y = \inf\{t > 0 : B_t = 0\}.$$

Let $M_t \in \mathrm{End}(T^*_{X_0}M, T^*_{X_t}M)$ be the solution to the following covariant SDE along the trajectory of L-diffusion X_t

$$\begin{aligned} \frac{\nabla}{\partial t} M_t &= -Ric(L)(X_t)M_t, \\ M_0 &= \mathrm{Id}_{T_{X_0}M}, \end{aligned}$$

where

$$\frac{\nabla}{\partial t} := U_t \frac{\partial}{\partial t} U_t^{-1}$$

denotes the Itô stochastic covariant derivative with respect to the Levi-Civita connection on M along $\{X_s, 0 \leq s \leq t\}$, $U_t \in \mathrm{End}(T^*_{X_0}M, T^*_{X_t}M)$ is the Itô stochastic parallel transport along $\{X_s, 0 \leq s \leq t\}$.

Probabilistic representation formula of Riesz transforms

Theorem (Li PTRF 2008)

Suppose that $Ric + \nabla^2\phi \geq -a$, *where* $a \geq 0$ *is a constant. Then*

$$\frac{1}{2}\nabla(a-L)^{-1/2}f(x) = \lim_{y\to+\infty} E_y\left[\int_0^\tau e^{a(s-\tau)}M_\tau M_s^{-1}\nabla e^{-B_s\sqrt{a-L}}f(X_s)dB_s \mid X_\tau = x\right].$$

In particular, if $Ric + \nabla^2\phi = -k$, *then* $\forall a \geq \max\{k, 0\}$,

$$\frac{1}{2}\nabla(a-L)^{-1/2}f(x) = \lim_{y\to+\infty} E_y\left[\int_0^\tau e^{(a-k)(s-\tau)}\nabla e^{-B_s\sqrt{a-L}}f(X_s)dB_s \mid X_\tau = x\right].$$

Example : Riesz transforms on $\mathbb{R}^n$

Let

$$M = \mathbb{R}^n, \ \phi = 0.$$

We have

$$L = \Delta, \quad Ric + \nabla^2 \phi = 0.$$

Thus, we can recapture the Gundy-Varopoulos representation formula of Riesz transforms on $\mathbb{R}^n$:

$$\frac{1}{2}\nabla(-\Delta)^{-1/2} f(x) = \lim_{y \to +\infty} E_y \left[\int_0^{\tau_y} \nabla e^{-B_s\sqrt{-\Delta}} f(X_s) dB_s \mid X_\tau = x \right].$$

Example : Riesz transforms on Gaussian spaces

Let

$$M = \mathbb{R}^n, \quad \phi = \frac{\|x\|^2}{2} + \frac{n}{2}\log(2\pi).$$

We have

- $L = \Delta - x \cdot \nabla$ is the Ornstein-Uhlenbeck operator on $\mathbb{R}^n$,
- $Ric + \nabla^2\phi = \mathrm{Id}$.

Thus we can recapture Gundy's representation formula (1986) for P.-A. Meyer's Riesz transform:

$$\begin{aligned} & \frac{1}{2}\nabla(a-L)^{-1/2}f(x) \\ = & \lim_{y\to+\infty} E_y\left[\int_0^{\tau_y} e^{(a+1)(s-\tau)}\nabla e^{-B_s\sqrt{a-L}}f(X_s)dB_s \,|\, X_\tau = x\right]. \end{aligned}$$

In particular, when $a = 0$, we get

$$\frac{1}{2}\nabla(-L)^{-1/2}f(x) = \lim_{y\to+\infty} E_y\left[\int_0^{\tau_y} e^{s-\tau}\nabla e^{-B_s\sqrt{-\Delta}}f(X_s)dB_s \,|\, X_\tau = x\right].$$

Example: Riesz transforms on Spheres

- Let $M = S^n$ and $\phi = 0$. We have $Ric = n - 1$. We then recapture Arcozzi's representation formula (1998)

$$\frac{1}{2}\nabla(a-\Delta_{S^n})^{-1/2}f(x) = \lim_{y\to+\infty} E_y\left[\int_0^{\tau_y} e^{(a+n-1)(s-\tau)}\nabla e^{-B_s\sqrt{a-\Delta_{S^n}}}f(X_s)dB_s \mid X_\tau = x\right.$$

- Let $M = S^n(\sqrt{n-1})$ and $\phi = 0$. We have $Ric = 1$. This leads to

$$\frac{1}{2}\nabla(a-\Delta_M)^{-1/2}f(x) = \lim_{y\to+\infty} E_y\left[\int_0^{\tau_y} e^{(a+1)(s-\tau)}\nabla e^{-B_s\sqrt{a-\Delta_M}}f(X_s)dB_s \mid X_\tau = x\right].$$

Example : P.-A. Meyer's Riesz transform on Wiener space

Taking $n \to \infty$ in the above formula, and using the Poincaré limit, we can obtain Gundy's representation formula on Wiener space (1986)

$$\frac{1}{2}\nabla(a-L)^{-1/2}f(x)$$
$$= \lim_{y\to+\infty} E_y\left[\int_0^{\tau_y} e^{(a+1)(s-\tau)}\nabla e^{-B_s\sqrt{a-L}}f(X_s)dB_s \mid X_\tau = x\right],$$

where

- L denotes the Ornstein-Uhlenbeck operator on Wiener space,
- X_t denotes the Ornstein-Uhlenbeck process on Wiener spacee.

Sharp estimates of Riesz transforms

In 1972, Pichorides proved that the L^p-norm of the Hilber transform is given by

$$\|H\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad \forall p > 1,$$

where

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

In 1996, Iwaniec and Martin extended the above result to Riesz transforms on $\mathbb{R}^n$. They proved that

$$\|R_j\|_{p,p} = \cot\left(\frac{\pi}{2p^*}\right), \quad \forall p > 1.$$

Using the Gundy-Varopoulos representation formula and the sharp L^p Burkholder inequality for martingale transforms, Bañuelos and Wang (Duke Math. J. 1995) gave a probabilistic proof of Iwaniec-Martin's result. Moreover, they proved that

$$\|R\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1,$$

where

$$R = \nabla(-\Delta)^{-1/2}.$$

Sharp L^p estimates of Riesz transforms on Riemannian manifolds

Based on our martingale representation for the Resz transform $\nabla(a-L)^{-1/2}$, and using the sharp L^p Burkholder inequality for subordinations of martingales, we have proved the following:

Theorem (Li PTRF 2008)

Suppose that $Ric(L) = Ric + \nabla^2\phi \geq 0$. Then, for all $p > 1$, we have

$$\|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1),$$

Suppose that $Ric(L) = Ric + \nabla^2\phi \geq -a$, where $a > 0$ is a constant. Then, for all $p > 1$, we have

$$\|\nabla(a-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1)(1 + 4\|T_1\|_p),$$

where, denoting B_t the Brownian motion on $\mathbb{R}^3$ with $B_0 = 0$,

$$T_1 := \inf\{t > 0 : \|B_t\| = 1\}.$$

Theorem (Li PTRF2008)

Let M be a complete Riemannian manifold with non-negative Ricci curvature. Then for all $1 < p < \infty$, we have

$$\|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1).$$

Moreover, at least in the Euclidean and Gaussian cases, the upper bound of type $O(p^* - 1)$ for the L^p norm of the Riesz transform $\nabla(-\Delta)^{-1/2}$ is asymptotically sharp when $p \to 1$ and $p \to \infty$.

Our results extend the estimates due to Iwaniec-Martin (Crelles96), Bañuelos-Wang (DMJ95)

$$\|\nabla(-\Delta_{\mathbb{R}^n})^{-/2}\|_{p,p} \leq 2(p^* - 1), \quad \forall p > 1.$$

Application : the Poincaré L^p-inequality

Theorem (Li PTRF2008)

Let $p > 1$, $q = \frac{p}{p-1}$. *Suppose that there exists a constant* $\rho > 0$ *such that*

$$\mathrm{Ric}(\mathrm{L}) = Ric + \nabla^2\phi \geq \rho.$$

Then for all $f \in C_0^\infty(M)$, *we have the following Poincaré inequality in* L^p

$$\|f - \mu(f)\|_p \leq \frac{\|\nabla(-L)^{-1/2}\|_{q,q}}{\sqrt{\rho}}\|\nabla f\|_p.$$

Equivalently,

$$\inf \sigma_p(-L) \setminus \{0\} > 0.$$

Note that: on all complete Riemannian manifold, we have

$$\|\nabla(-L)^{-1/2}\|_{2,2} = 1.$$

This leads us to recapture the famous Bakry-Emery criteria:

Theorem (Bakry-Emery 1985)

Suppose that there exists a constant $\rho > 0$ such that

$$\mathrm{Ric(L)} = Ric + \nabla^2\phi \geq \rho > 0.$$

Then for all $f \in C_0^\infty(M)$, we have

$$\|f - \mu(f)\|_2 \leq \frac{\|\nabla f\|_2}{\sqrt{\rho}}.$$

Equivalently,

$$\inf \sigma_2(-L) \setminus \{0\} > 0.$$

Riesz transforms on differential forms

In 1987, Bakry studied the Riesz transforms associated with the Hodge Laplacian acting on differential forms over complete Riemannian manifolds.

In 1990s, Shigekawa and Yoshida extended Bakry's results to Riesz transforms associated with the Hodge Laplacian acting on vector bundles over complete Riemannian manifolds.

In a paper which will appear in Revista Iberoamericana Mat., I established the martingale representation formula for the Riesz transforms acting on forms on complete Riemannian manifolds. This leads us to obtain the sharp and explicit L^p-bound of the Riesz transforms on forms.

Application: the L^p-estimates and existence theorems of d and $\bar{\partial}$

In two recent submitted papers, using the L^p-boundedness of the Riesz transforms and the Riesz potential, I established some global L^p-estimates and existence theorems of the Cartan-De Rham equation on complete Riemannian manifolds

$$d\omega = \alpha, \quad d\alpha = 0$$

and the Cauchy-Riemann equation on complete Kähler manifolds

$$\bar{\partial}\omega = \alpha, \quad \bar{\partial}\alpha = 0.$$

Theorem (Li 2009)

Let M be a complete Riemannian manifold, $\phi \in C^2(M)$ and $d\mu = e^{-\phi} dv$. Suppose that there exists a constant $\rho > 0$ such that

$$W_k + \nabla^2 \phi \geq \rho,$$

and

$$W_{k-1} + \nabla^2 \phi \geq 0,$$

where W_k denotes the Weitzenböck curvature operator on k-forms. Then, for all $\alpha \in L^p(\Lambda^k T^* M, \mu)$ such that $d\alpha = 0$, there exists some $\omega \in L^p(\Lambda^{k-1} T^* M, \mu)$ such that

$$d\omega = \alpha,$$

and satisfying

$$\|\omega\|_p \leq \frac{C_k (p^* - 1)^{3/2}}{\sqrt{\rho}} \|\alpha\|_p.$$

Thank you!