Optimal Investment, Consumption and Retirement Decision with Disutility and Borrowing Constraints

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Merton (1969 RESTAT, 1971 JET)

Formulate a continuous time consumption/investment problem. (dynamic programming method)

\[
\max_{c, \pi} \mathbb{E} \left[ \int_0^\infty e^{-\beta t} u(c_t) dt \right]
\]

subject to

\[
dX_t = [rX_t + \pi_t(\mu - r) - c_t] dt + \pi_t \sigma dB_t.
\]
Karatzas and Wang (2000 SICON)
Consider the mixture problem $(c, \pi, \tau)$. (martingale method)

$$\max_{c, \pi, \tau} \mathbb{E} \left[ \int_0^\tau e^{-\beta t} u_1(c_t) \, dt + e^{-\beta \tau} u_2(X_\tau) \right]$$

subject to

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t] \, dt + \pi_t \sigma \, dB_t.$$
Portfolio Selection (3)

Choi and Shim (2006 MF), Benchmark Model

Labor income, disutility and optimal retirement time. (dynamic programming without considering borrowing constraints)

\[
\max_{c, \pi, \tau} E \left[ \int_0^\infty \left( u(c_t) - l \mathbf{1}_{\{0 \leq t < \tau\}} \right) dt \right]
\]

subject to

\[
dX_t = [rX_t + \pi_t(\mu - r) - c_t + \epsilon \mathbf{1}_{\{0 \leq t < \tau\}}] dt + \pi_t \sigma dB_t.
\]
Borrowing Constraints

Borrowing Constraint (Liquidity Constraint)

The economic agent has limited opportunities to borrow against future labor income and cannot totally insure the risk of income fluctuation. So the borrowing constraint restricts the agent’s choice in a non-trivial way.

Literatures

Borrowing Constraints

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Literatures

The Basic Financial Market Setup (1)

- **riskless asset** $S_0(t)$: \[ \frac{dS_0(t)}{S_0(t)} = r dt \]
- **risky asset** $S_t$: \[ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t \]

- $r, \mu, \sigma$: constants
- $B_t$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- **market-price-of-risk** $\theta \triangleq \frac{\mu - r}{\sigma}$
- **discount process** $\zeta_t \triangleq \exp\{-rt\}$
- **exponential martingale** $Z_t \triangleq \exp\{-\theta B_t - \frac{1}{2} \theta^2 t\}$
- **pricing kernel**(or state-price-density) $H_t \triangleq \zeta_t Z_t$
- **equivalent martingale measure**
  \[ \tilde{\mathbb{P}}^T(A) \triangleq \mathbb{E}[Z_T 1_A], \text{ for any fixed } T \in [0, \infty) \text{ and for } A \in \mathcal{F}_T \]
  \[ \tilde{B}_t^T \triangleq B_t + \theta t : \text{a standard Brownian motion under the new measure } \tilde{\mathbb{P}}^T \text{ by Girsanov theorem} \]
  \[ \exists \tilde{\mathbb{P}} \text{ on } \mathcal{F}_\infty, \text{ which agrees with } \tilde{\mathbb{P}}^T \text{ on } \mathcal{F}_T \text{ for any } T \in [0, \infty). \text{ Furthermore } \tilde{B}_t \text{ is a standard Brownian motion under } \tilde{\mathbb{P}}. \]
The Basic Financial Market Setup (2)

- labor wage: $\epsilon > 0$, disutility: $l > 0$
- retirement time: a stopping time $\tau$
- consumption process: $c_t$ with $\int_0^t c_s ds < \infty$, a.s.
- portfolio process: $\pi_t$ with $\int_0^t \pi_s^2 ds < \infty$, a.s.
- wealth process $X_t$ with an initial wealth $X_0 = x \geq 0$

$$dX_t = \left[rx_t + \pi_t(\mu - r) - c_t + \epsilon 1_{\{0 \leq t < \tau\}}\right]dt + \pi_t \sigma dB_t$$

$$= \left[rx_t - c_t + \epsilon 1_{\{0 \leq t < \tau\}}\right]dt + \pi_t \sigma d\tilde{B}_t$$

budget constraint

$$\mathbb{E} \left[ H_{\tau} X_{\tau} + \int_0^\tau H_s c_s ds - \int_0^\tau H_s \epsilon ds \right] \leq x, \text{ for all } \tau$$
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- wealth process $X_t$ with an initial wealth $X_0 = x \geq 0$

$$
\begin{align*}
    dX_t &= \left[rx_t + \pi_t(\mu - r) - c_t + \epsilon 1_{\{0 \leq t < \tau\}} \right] dt + \pi_t \sigma dB_t \\
    &= \left[rx_t - c_t + \epsilon 1_{\{0 \leq t < \tau\}} \right] dt + \pi_t \sigma d\tilde{B}_t
\end{align*}
$$

budget constraint

$$
\mathbb{E} \left[ H_\tau X_\tau + \int_0^\tau H_s c_s ds - \int_0^\tau H_s \epsilon ds \right] \leq x, \text{ for all } \tau
$$
The borrowing constraint means that the investor cannot borrow against her/his future labor income. So the wealth of the investor should always be nonnegative. i.e. \( X_t \geq 0, \ \forall t > 0 \)

Borrowing Constraint

\[
\mathbb{E} \left[ H_T X_T + \int_t^T H_s (c_s - \epsilon) ds \middle| \mathcal{F}_t \right] \geq 0, \text{ for all } 0 \leq t < \tau.
\]
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Borrowing Constraint

$$\mathbb{E} \left[ H_{\tau} X_{\tau} + \int_{t}^{\tau} H_s (c_s - \epsilon) ds \bigg| \mathcal{F}_t \right] \geq 0, \ \text{for all } 0 \leq t < \tau.$$
Definition 1 (General Utility Function)

A function \( u : (0, \infty) \rightarrow \mathbb{R} \) is a utility function if it is strictly increasing, strictly concave, continuously differentiable and satisfies

\[
    u'(0+) \triangleq \lim_{c \downarrow 0} u'(c) = \infty, \quad u'(\infty) \triangleq \lim_{c \uparrow \infty} u'(c) = 0.
\]

Labor Income

\( \epsilon \): the agent receives the income continuously

Disutility

\( l \): disutility comes from labor

Retirement Time

\( \tau \): the immortal agent can choose when to retire
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Expected Utility Maximization Problem

The immortal investor wants to maximize her/his expected utility:

$$J(x; c, \pi, \tau) \triangleq \mathbb{E} \left[ \int_0^\infty e^{-\beta t} (u(c_t) - l1_{\{0 \leq t < \tau\}}) \, dt \right]$$

i.e.,

$$V(x) = \sup_{(c, \pi, \tau) \in A(x)} J(x; c, \pi, \tau)$$

subject to the budget constraint and the borrowing constraint, where $A(x)$ is the set of an admissible triple $(c, \pi, \tau)$ and $\beta > 0$ is the subjective discount rate.
Expected Utility Maximization Problem

The investor wants to maximize her/his expected utility

$$
\sup_{(c, \pi, \tau) \in A(x)} J(x; c, \pi, \tau) \triangleq \sup_{(c, \pi, \tau) \in A(x)} \mathbb{E} \left[ \int_0^\infty e^{-\beta t} \left( u(c_t) - l \mathbb{1}_{\{0 \leq t < \tau\}} \right) dt \right]
$$

$$
= \sup_{(c, \pi, \tau) \in A(x)} \mathbb{E} \left[ \int_0^\tau e^{-\beta t} (u(c_t) - l) dt + e^{-\beta \tau} U(X_\tau) \right],
$$

subject to the budget constraint

$$
\mathbb{E} \left[ \int_0^\tau H_s c_s ds + H_\tau X_\tau - \int_0^\tau H_s \epsilon ds \right] \leq x,
$$

and the borrowing constraint,

$$
\mathbb{E} \left[ H_\tau X_\tau + \int_t^\tau H_s (c_s - \epsilon) ds \Bigg| \mathcal{F}_t \right] \geq 0.
$$
Lemma 2 (Value Function After Retirement)

The value function $U(\cdot)$ is given by

$$U(x) = \frac{2(\lambda^{**})^{n_+}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^{**}} \frac{zl_1(z) - u(l_1(z))}{z^{n_+ + 1}} dz$$

$$- \frac{2(\lambda^{**})^{n_-}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^{**}} \frac{zl_1(z) - u(l_1(z))}{z^{n_- + 1}} dz + (\lambda^{**})x,$$

where $\hat{y} > 0$ is an arbitrary constant, $l_1(\cdot)$ is the inverse function of $u'(\cdot)$ and $\lambda^{**}$ is determined by the algebraic equation

$$- \frac{2n_+(\lambda^{**})^{n_+ - 1}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^{**}} \frac{zl_1(z) - u(l_1(z))}{z^{n_+ + 1}} dz$$

$$+ \frac{2n_- (\lambda^{**})^{n_- - 1}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^{**}} \frac{zl_1(z) - u(l_1(z))}{z^{n_- + 1}} dz = x.$$

Here $n_+ > 1$ and $n_- < 0$ are two roots of the following quadratic equation

$$\frac{1}{2} \theta^2 n^2 + \left( \beta - r - \frac{1}{2} \theta^2 \right) n - \beta = 0. \quad (1)$$
The Value Function

The value function of our problem is given by

$$V(x) = \sup_{(c, \pi, \tau) \in A(x)} J(x; c, \pi, \tau)$$

$$= \sup_{\tau} \sup_{(c, \pi) \in \Pi_{\tau}(x)} J(x; c, \pi, \tau)$$

$$\triangleq \sup_{\tau} V_{\tau}(x)$$

where $A(x)$ is the set of an admissible triple $(c, \pi, \tau)$ and $\Pi_{\tau}(x)$ is the set of $\tau$-fixed consumption-portfolio plan $(c, \pi)$ for which $(c, \pi, \tau) \in A(x)$.
Individual’s Shadow Prices Problem (He and Pagés (1993))

\[
\inf_{D_t > 0} \tilde{J}(\lambda, D_t; \tau) \triangleq \inf_{D_t > 0} \mathbb{E} \left[ \int_0^\tau e^{-\beta t} \left\{ \tilde{u}(y_t^\lambda) + y_t^\lambda \epsilon - l \right\} \, dt + e^{-\beta \tau} \tilde{U}(y_\tau^\lambda) \right],
\]

where \( D_t \) is the non-negative, decreasing, and progressively measurable process, \( y_t^\lambda = \lambda D_t e^{\beta t} H_t \)

\[
\tilde{u}(y) \triangleq \sup_c \{ u(c) - cy \} = u(l_1(y)) - y l_1(y)
\]

\[
\tilde{U}(y) \triangleq \sup_x \{ U(x) - xy \} = U(l_2(y)) - y l_2(y),
\]

where \( l_1(\cdot) \triangleq u'(\cdot)^{-1} \) and \( l_2(\cdot) \triangleq U'(\cdot)^{-1} \). Moreover \( \tilde{u}(\cdot) \) and \( \tilde{U}(\cdot) \) are strictly decreasing, strictly convex.
Duality Approaches (2)

\[ J(x; \mathbf{c}, \pi, \tau) = \mathbb{E} \left[ \int_0^T e^{-\beta t} \{ u(c_t) - l - \lambda D_t e^{\beta t} H_t c_t \} dt \right] \]
\[ + e^{-\beta \tau} \{ U(X_{\tau}) - \lambda D_{\tau} e^{\beta \tau} H_{\tau} X_{\tau} \} + \lambda \mathbb{E} \left[ \int_0^\tau D_t H_t c_t dt + D_{\tau} H_{\tau} X_{\tau} \right] \]
\[ \leq \mathbb{E} \left[ \int_0^T e^{-\beta t} \tilde{u}(\lambda D_t e^{\beta t} H_t) dt + e^{-\beta \tau} \tilde{U}(\lambda D_{\tau} e^{\beta \tau} H_{\tau}) - \int_0^T e^{-\beta t} l dt \right] \]
\[ + \lambda \mathbb{E} \left[ \int_0^\tau D_t H_t c_t dt + D_{\tau} H_{\tau} X_{\tau} \right] \]
\[ \leq \mathbb{E} \left[ \int_0^T e^{-\beta t} \tilde{u}(\lambda D_t e^{\beta t} H_t) dt + e^{-\beta \tau} \tilde{U}(\lambda D_{\tau} e^{\beta \tau} H_{\tau}) - \int_0^T e^{-\beta t} l dt \right] \]
\[ + \mathbb{E} \left[ \int_0^\tau \lambda D_t H_t \epsilon dt \right] + \lambda x \]
\[ = \mathbb{E} \left[ \int_0^T e^{-\beta t} \left\{ \tilde{u}(\lambda D_t e^{\beta t} H_t) dt + \lambda D_t e^{\beta t} H_t \epsilon - l \right\} dt \right] \]
\[ + e^{-\beta \tau} \tilde{U}(\lambda D_{\tau} e^{\beta \tau} H_{\tau}) \]
Duality Approaches (3)

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T D_t H_t c_t dt + D_T H_T X_T \right] \\
= \mathbb{E} \left[ \int_0^T D_t H_t (c_t - \epsilon) dt + D_T H_T X_T + \int_0^T D_t H_t \epsilon dt \right] \\
= \mathbb{E} \left[ \int_0^T D_t H_t \epsilon dt + \int_0^T H_t c_t dt - \int_0^T H_t \epsilon dt + H_T X_T \right] \\
+ \mathbb{E} \left[ \int_0^T \mathbb{E} \left[ \int_t^T H_s c_s ds + H_T X_T - \int_t^T H_s \epsilon ds \bigg| \mathcal{F}_t \right] dD_t \right] \\
\leq \mathbb{E} \left[ \int_0^T D_t H_t \epsilon dt \right] + x.
\end{align*}
\]
Duality Approaches (4)

For any fixed $\tau \in S$, previous inequalities hold as equality if

$$c_t = l_1(\lambda D_t e^{\beta t} H_t), \quad X_\tau = l_2(\lambda D_\tau e^{\beta \tau} H_\tau), \text{ for all } 0 \leq t \leq \tau,$$

$$\mathbb{E} \left[ \int_0^\tau H_t c_t dt + H_\tau X_\tau - \int_0^\tau H_t \epsilon dt \right] = x,$$

and

$$\mathbb{E} \left[ \int_t^\tau H_s c_s ds + H_\tau X_\tau - \int_t^\tau H_s \epsilon ds \bigg| \mathcal{F}_t \right] = 0. \quad (2)$$
Duality Approaches (5)

(Based on Karatzas and Wang (2000))

\[ V(x) = \sup_{\tau \in S} V_{\tau}(x) = \sup_{\tau \in S} \inf_{\{\lambda > 0, D_t > 0\}} \left[ \tilde{J}(\lambda, D_t; \tau) + \lambda x \right] \]
\[ = \inf_{\{\lambda > 0, D_t > 0\}} \sup_{\tau \in S} \left[ \tilde{J}(\lambda, D_t; \tau) + \lambda x \right], \]

**Proposition (Value Function)**

Define

\[ \tilde{V}(\lambda) \triangleq \sup_{\tau \in S} \inf_{D_t > 0} \tilde{J}(\lambda, D_t; \tau) = \inf_{D_t > 0} \sup_{\tau \in S} \tilde{J}(\lambda, D_t; \tau), \]

then if \( \tilde{V}(\lambda) \) exists and is differentiable for \( \lambda > 0 \), then

\[ V(x) = \inf_{\lambda > 0} \left[ \tilde{V}(\lambda) + \lambda x \right], \]

for any \( x \in (0, \infty) \).
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then if \( \tilde{V}(\lambda) \) exists and is differentiable for \( \lambda > 0 \), then

\[ V(x) = \inf_{\lambda > 0} \left[ \tilde{V}(\lambda) + \lambda x \right], \]

for any \( x \in (0, \infty) \).
To find $\tilde{V}(\lambda)$, define

$$\phi(t, y) \triangleq \sup_{\tau > t} \inf_{D_t > 0} \mathbb{E}^{y_t = y} \left[ \int_t^\tau e^{-\beta s} \{\tilde{u}(y_s) + \epsilon y_s - l\} \, ds + e^{-\beta \tau} \tilde{U}(y_\tau) \right],$$

where $y_t = \lambda D_t e^{\beta t} H_t$, $y_0 = \lambda > 0$. Then

$$\frac{dy_t}{y_t} = \frac{dD_t}{D_t} + (\beta - r) dt - \theta dB_t.$$

$\phi(0, \lambda) = \tilde{V}(\lambda)$.

This optimal stopping problem can be solved by the variational inequality.
Suppose $D_t$ has a following differential form $dD_t = -\psi(t)D_t dt$ for some $\psi(t) \geq 0$.

The Bellman equation is given by

$$\min \left\{ \mathcal{L}\phi(t, y) + e^{-\beta t} \{\tilde{u}(y) + \epsilon y - l\}, -\frac{\partial \phi}{\partial y} \right\} = 0$$

with the differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + (\beta - r)y\frac{\partial}{\partial y} + \frac{1}{2}\theta^2 y^2 \frac{\partial^2}{\partial y^2}.$$ 

(He and Pagés (1993))
Variational Inequality 2.1

Find the free boundary $\bar{y}, \hat{y}$ which makes zero wealth level and a function $\tilde{\phi}(\cdot, \cdot) \in C^1((0, \infty) \times \mathbb{R}^+) \cap C^2((0, \infty) \times (\mathbb{R}^+ \setminus \{\bar{y}\}))$ satisfying

1. $\mathcal{L}\tilde{\phi} + e^{-\beta t}\{\tilde{u}(y) + \epsilon y - l\} = 0$, $\bar{y} < y \leq \hat{y}$
2. $\mathcal{L}\tilde{\phi} + e^{-\beta t}\{\tilde{u}(y) + \epsilon y - l\} \leq 0$, $0 < y \leq \bar{y}$
3. $\tilde{\phi}(t, y) > e^{-\beta t}\tilde{U}(y)$, $y > \bar{y}$
4. $\tilde{\phi}(t, y) = e^{-\beta t}\tilde{U}(y)$, $0 < y \leq \bar{y}$
5. $\frac{\partial \tilde{\phi}}{\partial y}(t, y) \leq 0$, $0 < y \leq \hat{y}$
6. $\frac{\partial \tilde{\phi}}{\partial y}(t, y) = 0$, $y \geq \hat{y}$

for all $t > 0$, with boundary conditions

$$\frac{\partial \tilde{\phi}}{\partial y}(t, \hat{y}) = 0 \quad \text{and} \quad \frac{\partial^2 \tilde{\phi}}{\partial y^2}(t, \hat{y}) = 0.$$
• \exists \text{ one-to-one correspondence between } y \text{ and } x.
∃ one-to-one correspondence between $y$ and $x$.

$$e^{-\beta t} \tilde{U}(y) \quad \tilde{L}_{\tilde{\phi}} + e^{-\beta t} \{ \tilde{u}(y) + \epsilon y - l \} = 0 \quad \frac{\partial \tilde{\phi}}{\partial y}(t, y) = 0$$
\( \exists \) one-to-one correspondence between \( y \) and \( x \).

\[
e^{-\beta t} \tilde{U}(y) + L\tilde{\phi} + e^{-\beta t} \{ \tilde{u}(y) + \epsilon y - l \} = 0
\]

\[
\frac{\partial \tilde{\phi}}{\partial y}(t, y) = 0
\]
Proposition 2

Consider the function

\[ v(y) = \begin{cases} 
C_1 y^{n_+} + C_2 y^{n_-} 
+ \frac{y}{\theta^2 (n_+ - n_-)} \int_{\hat{y}}^y \frac{l(z) - u(l(z))}{z^{n_+ + 1}} dz 
- \frac{2y}{\theta^2 (n_+ - n_-)} \int_{\hat{y}}^y \frac{l(z) - u(l(z))}{z^{n_- + 1}} dz, & \text{if } \bar{y} < y \leq \hat{y}, \\
\frac{2y^{n_+}}{\theta^2 (n_+ - n_-)} \int_{\hat{y}}^y \frac{l(z)}{z^{n_+ + 1}} dz 
- \frac{2y^{n_-}}{\theta^2 (n_+ - n_-)} \int_{\hat{y}}^y \frac{l(z)}{z^{n_- + 1}} dz, & \text{if } 0 < y \leq \bar{y},
\end{cases} \]

then \( \tilde{\phi}(t, y) = e^{-\beta t} v(y) \) is a solution to Variational Inequality. And the coefficients \( C_1, C_2, \hat{y} \) and the free boundary value \( \bar{y} \) are determined implicitly.

\[ \tilde{V}(\lambda) = \tilde{\phi}(0, \lambda) = v(\lambda) \]
Value Function

Theorem 3

The value function $V(x)$ is given by

$$V(x) = \begin{cases} 
C_1(\lambda^*)^{n_+} + C_2(\lambda^*)^{n_-} + (\lambda^*)x \\
+ \frac{2(\lambda^*)^{n_+}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^*} \frac{l + z(l_1(z) - \epsilon) - u(l_1(z))}{z^{n_+ + 1}} dz \\
- \frac{2(\lambda^*)^{n_-}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^*} \frac{l + z(l_1(z) - \epsilon) - u(l_1(z))}{z^{n_- + 1}} dz, & \text{if } 0 \leq x < \bar{x}, \\
U(x), & \text{if } x \geq \bar{x} 
\end{cases}$$

where

$$\bar{x} = l_2(\bar{y}),$$

where $\lambda^*$ is determined from the following algebraic equation

$$-n_+ C_1(\lambda^*)^{n_+ - 1} - n_- C_2(\lambda^*)^{n_- - 1}$$

$$- \frac{2n_+(\lambda^*)^{n_+ - 1}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^*} \frac{l + z(l_1(z) - \epsilon) - u(l_1(z))}{z^{n_+ + 1}} dz$$

$$+ \frac{2n_-(\lambda^*)^{n_- - 1}}{\theta^2(n_+ - n_-)} \int_{\hat{y}}^{\lambda^*} \frac{l + z(l_1(z) - \epsilon) - u(l_1(z))}{z^{n_- + 1}} dz = x$$
Optimal Wealth Processes

Before Retirement

\[ X^*(t) = -n_+ C_1 (y_t^{\lambda^*})^{n_+ - 1} - n_- C_2 (y_t^{\lambda^*})^{n_- - 1} \]
\[ \quad - \frac{2n_+ (y_t^{\lambda^*})^{n_+ - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{y}} y_t^{\lambda^*} \frac{l + z(l_1(z) - \epsilon) - u(l_1(z))}{z^{n_+ + 1}} dz \]
\[ + \frac{2n_- (y_t^{\lambda^*})^{n_- - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{y}} y_t^{\lambda^*} \frac{l + z(l_1(z) - \epsilon) - u(l_1(z))}{z^{n_- + 1}} dz \]

After Retirement

\[ X^{**}(t) = - \frac{2n_+ (y_t^{\lambda^{**}})^{n_+ - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{y}} y_t^{\lambda^{**}} \frac{zl_1(z) - u(l_1(z))}{z^{n_+ + 1}} dz \]
\[ + \frac{2n_- (y_t^{\lambda^{**}})^{n_- - 1}}{\theta^2 (n_+ - n_-)} \int_{\hat{y}} y_t^{\lambda^{**}} \frac{zl_1(z) - u(l_1(z))}{z^{n_- + 1}} dz \]
Theorem 4

The optimal policies \((c^*, \pi^*, \tau^*)\) are given by

\[
c^*_t = \begin{cases} 
    l_1(y^*_t), & \text{if } 0 \leq X_t < \bar{x} \\
    l_1(y^{**}_t), & \text{if } X_t \geq \bar{x},
\end{cases}
\]

With the optimal wealth process \(X^*(t)\), the optimal stopping time \(\tau^*\) is determined by

\[
\tau^* = \inf \{ t > 0 \mid X^*(t) \geq \bar{x} \}.
\]
Optimal Policies (2)

Theorem 5

(Continued)

\[
\pi^*_t = \begin{cases} 
\theta \left\{ \frac{n_+ (n_+ - 1) C_1 (y_t^\lambda)^{n_+ - 1} + n_- (n_- - 1) C_2 (y_t^\lambda)^{n_- - 1}}{\sigma^2} + \frac{2}{\theta^2} \frac{l+y_t^\lambda (l_1 (y_t^\lambda) - \epsilon) - u(l_1 (y_t^\lambda))}{y_t^\lambda} ight. \\
+ \frac{2n_+ (n_+ - 1) (y_t^\lambda)^{n_+ - 1}}{\theta^2 (n_+ - n_-)} \int_{y_t^\lambda} \frac{l+y_t^\lambda (l_1 (z) - \epsilon) - u(l_1 (z))}{z^{n_+ + 1}} dz & \text{if } 0 \leq X_t < \bar{x} \\
\frac{2}{\sigma \theta} \left\{ \frac{y_t^\lambda** l_1 (y_t^\lambda**)}{y_t^\lambda**} - u(l_1 (y_t^\lambda**)) ight. \\
+ \frac{n_+ (n_+ - 1) (y_t^\lambda**)^{n_+ - 1}}{n_+ - n_-} \int_{y_t^\lambda**} \frac{l+y_t^\lambda** (l_1 (z) - \epsilon) - u(l_1 (z))}{z^{n_+ + 1}} dz & \text{if } X_t \geq \bar{x}. \\
- \frac{n_- (n_- - 1) (y_t^\lambda**)^{n_- - 1}}{n_+ - n_-} \int_{y_t^\lambda**} \frac{l+y_t^\lambda** (l_1 (z) - \epsilon) - u(l_1 (z))}{z^{n_- + 1}} dz \\
\left. \right\} 
\end{cases}
\]
Examples: CRRA Utility Class

Definition 6 (CRRA Utility Function)
A CRRA utility function is defined by

\[ u(c) \triangleq \begin{cases} 
\frac{1}{1-\gamma} c^{1-\gamma}, & \text{if } \gamma > 0 \text{ and } \gamma \neq 1, \\
\log c, & \text{if } \gamma = 1.
\end{cases} \]

Here \( \gamma \) is an investor’s coefficient of relative risk aversion.

Merton’s Constant

\[ K \triangleq r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \theta^2 > 0. \]
Expected Utility Maximization Problem (Power-Type)

\[ J(x; c, \pi, \tau) = \mathbb{E} \left[ \int_0^\tau e^{-\beta t} \left( \frac{1}{1-\gamma} c^{1-\gamma} - l \right) dt + \int_\tau^\infty e^{-\beta t} \frac{1}{1-\gamma} c^{1-\gamma} dt \right] \]

\[ = \mathbb{E} \left[ \int_0^\tau e^{-\beta t} \left( \frac{1}{1-\gamma} c^{1-\gamma} - l \right) dt + e^{-\beta \tau} U(X_\tau) \right] \]

Required Functions

\[ U(x) = \frac{1}{K^\gamma} \frac{1}{1-\gamma} x^{1-\gamma} \]

\[ \tilde{u}(y) = \frac{\gamma}{1-\gamma} y^{-\frac{1-\gamma}{\gamma}} \]

\[ \tilde{U}(y) = \frac{\gamma}{K(1-\gamma)} y^{-\frac{1-\gamma}{\gamma}} \]
Expected Utility Maximization Problem (Power-Type)

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Required Functions

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\[ \tilde{U}(y) = \frac{\gamma}{K(1-\gamma)} y^{-\frac{1-\gamma}{\gamma}} \]
Proposition (Value Function)
From Theorem 3,

\[
V(x) = \begin{cases} 
  c_1(\lambda^*)^{n_+} + c_2(\lambda^*)^{n_-} + \frac{\gamma}{K(1-\gamma)}(\lambda^*)^{-\frac{1-\gamma}{\gamma}} \\
  + (x + \frac{\epsilon}{r})(\lambda^*) - \frac{1}{\beta}, & \text{if } 0 \leq x < \bar{x} \\
  \frac{1}{K\gamma} \frac{1}{1-\gamma} x^{1-\gamma}, & \text{if } x \geq \bar{x}
\end{cases}
\]

where \( \lambda^* \) is determined from the algebraic equation

\[
-n_+ c_1(\lambda^*)^{n_+ - 1} - n_- c_2(\lambda^*)^{n_- - 1} + \frac{1}{K}(\lambda^*)^{-\frac{1}{\gamma}} - \frac{\epsilon}{r} = x, \text{ for } 0 \leq x < \bar{x}
\]

and the critical wealth level is given by \( \bar{x} = \frac{1}{K} \bar{y}^{-\frac{1}{\gamma}} \).
CRRA Utility Class - Optimal Policies

Optimal Policies

\[ c_t^* = \begin{cases} (y_t^{*})^{-\frac{1}{\gamma}}, & \text{if } 0 \leq X_t < \bar{x} \\ KX_t, & \text{if } X_t \geq \bar{x}, \end{cases} \]

\[ \pi_t^* = \begin{cases} \frac{\theta}{\sigma} \left\{ n_+(n_+ - 1)c_1(y_t^{*})^{n_+ - 1} \\ + n_-(n_- - 1)c_2(y_t^{*})^{n_- - 1} + \frac{1}{K\gamma} (y_t^{*})^{-\frac{1}{\gamma}} \right\}, & \text{if } 0 \leq X_t < \bar{x} \\ \frac{\theta}{\sigma\gamma} X_t, & \text{if } X_t \geq \bar{x}, \end{cases} \]

\[ \tau^* = \inf \{ t > 0 \mid X^*(t) \geq \bar{x} \}, \]

where the optimal wealth process before retirement is given by

\[ X^*(t) = -n_+ c_1(y_t^{*})^{n_+ - 1} - n_- c_2(y_t^{*})^{n_- - 1} + \frac{1}{K} (y_t^{*})^{-\frac{1}{\gamma}} - \frac{\epsilon}{r}. \]
Numerical Results for a CRRA Utility Function (1)

**Figure 1:** Comparison of amount of wealth invested in the risky asset ($\beta = 0.07$, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.2$, $\gamma = 2$, $\epsilon = 0.2$ and $l = 0.5$)
Numerical Results for a CRRA Utility Function (2)

**Figure 2:** Comparison of consumption ratio
($\beta = 0.07, \ r = 0.01, \ \mu = 0.05, \ \sigma = 0.2, \ \gamma = 2, \ \epsilon = 0.2 \ and \ l = 0.5$)
Conclusion

- We extended the optimal consumption-portfolio selection problem of an infinitely-lived working investor whose wealth is subject to borrowing constraint to the general utility function case.

- We figured out that the critical wealth level with borrowing constraint is lower than the level with no constraint for the CRRA utility case.

- The amount of investing to risky asset with borrowing constraint is lower than the amount with no constraint for the CRRA utility case.
Thank you!