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On the Complexity of a Class of Discrete Fixed Point Problems under the Lexicographic Ordering*

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Abstract

Let $C = \{x \in \mathbb{Z}^n \mid a \leq x \leq b\}$ with $a \leq b$ being two finite integer points and $f(x)$ be an increasing mapping in terms of the lexicographic ordering from C to itself with $f(a) = a$ and $f(b) \neq b$.

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This paper is concerned with the complexity of a class of discrete fixed point problems: Compute a fixed point of $f(x)$ in C other than a when the existence is assured under some conditions. This class of discrete fixed point problems embraces the Tarski's fixed point problem under the componentwise ordering and the well-known equal-sums problem. We prove in this paper that this entire class of discrete fixed point problems is in a new class of complexity named as Bipartie PPAD, which has a PPAD graph but with two known nodes as end nodes of two different paths. To further demonstrate the hardness, we also give a reduction of the problem of Nash equilibria of a bimatrix game to the class of discrete fixed point problems.

Keywords: Lattice, Lexicographic Ordering, Componentwise Ordering, Increasing Mapping, Fixed Point, PPAD, Bipartie PPAD

1 Introduction

The fixed-point computation remains a key component in optimization and machine learning fields, with applications from Sequential Game, Markov Decision Processes, and Q-Learning; see, e.g., Shapley [10], Bellman [1] and Kearns and Singh [7].

In this paper we consider the complexity of a general class of discrete fixed point problems, which include Tarski's fixed point problem, the equal-sums problem, etc.. Specifically, let $\preceq$ be a binary relation on a nonempty set S. The pair $(S, \preceq)$ is a partially ordered set if $\preceq$ is reflexive, transitive and antisymmetric. A lattice is a partially ordered set $(S, \preceq)$, in which any two elements x and y have a least upper bound (supremum), $\sup_S(x, y) = \inf\{z \in S \mid x \preceq z \text{ and } y \preceq z\}$, and a greatest lower bound (infimum), $\inf_S(x, y) = \sup\{z \in S \mid z \preceq x \text{ and } z \preceq y\}$, in the set. A lattice $(S, \preceq)$ is complete if every nonempty subset of S has a supremum and an infimum in S. Let $g(x)$ be a mapping from S to itself. We say g is an increasing mapping if $g(x) \preceq g(y)$ for any x and y of S with $x \preceq y$.

Let $N = \{1, 2, \ldots, n\}$. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ of $\mathbb{R}^n$, $x \leq y$ if $x_i \leq y_i$ for all $i \in N$, which is the componentwise ordering, and $x \leq_\ell y$ if either $x = y$ or $x_i = y_i$ for all $i < j$ and $x_j < y_j$ for some $j \in N$, which is the lexicographic ordering. Let

$$C = \{x \in \mathbb{Z}^n \mid a \leq x \leq b\},$$

where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are two finite integer points with $a \leq b$. Let $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ be an increasing mapping in terms of the lexicographic ordering from C to itself such that $f(a) = a$ and $f(b) \neq b$. We are concerned with the **complexity of a class of discrete fixed point problems: Compute a fixed point of $f(x)$ in C other than** a when the existence is assured under some condtions. This class of discrete fixed point problems includes Tarski's fixed point problem under the componentwise ordering in Tarski (1955) and the equal-sums problem in Papadimitriou (1994).

The well-known Tarski's fixed point theorem is as follows.

Theorem (Tarski, 1955) If $(S, \preceq)$ is a complete lattice and $g(x)$ is an increasing mapping from S to itself, then there exists $x^* \in S$ such that $g(x^*) = x^*$, which is a fixed point of $g(x)$.

Note that Tarski's fixed point theorem is significantly different from the classical Brouwer, Sperner lemma, or Kakutani's fixed point theorems where the mapping $g(x)$ is assumed to be continuous or semi-continuous. Tarski's fixed point theorem does not deal with continuous functions, and it simply states that any order-preserving function on a complete lattice has a fixed point, and indeed a smallest fixed point and a largest fixed point.

Let $\Pi = \{x \in \mathbb{Z}^n \mid c \leq x \leq d\}$, where $c = (c_1, c_2, \ldots, c_n)$ and $d = (d_1, d_2, \ldots, d_n)$ are two finite integer points with $c \leq d$. Under the lexicographic ordering, Π is a complete lattice. Let $f(x)$ be an increasing mapping in terms of the lexicographic ordering from Π to itself. Tarski's fixed point theorem shows that $f(x)$ has a fixed point in Π. The Tarski's fixed point problem under the lexicographic ordering is: Compute a fixed point of $f(x)$ in Π. Let $C = \{x \in \mathbb{Z}^n \mid a \leq x \leq b\}$, where $a = c - w$ and $b = d + w$ with

$0 \leq w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n$ being any given nonzero integer vector. For $x \in C \backslash \Pi$, we expand f by simply setting

$$f(x) = \begin{cases} f(y) & \text{if } c \leq_\ell x, \\ a & \text{if } x \leq_\ell c, \end{cases}$$

where y is the largest point in Π such that $y \leq_\ell x$. With this expansion, $f(x)$ becomes an increasing mapping in terms of the lexicographic ordering from C to itself with $f(a) = a$ and $f(b) \neq b$. Thus, **the Tarski's fixed point problem under the lexicographic ordering can be reduced to the class of discrete fixed point problems**. A preliminary study on the complexity of Tarski's fixed point theorem was given in Chang et al. (2008).

Let $h(x)$ be an increasing mapping in terms of the componentwise ordering from Π to itself. Under the componentwise ordering, Π is a complete lattice. Tarski's fixed point theorem asserts that $h(x)$ has a fixed point in Π. The Tarski's fixed point problem under the componentwise ordering is: Compute a fixed point of $h(x)$ in Π. For $x \in C$, we define

$$f(x) = \begin{cases} x & \text{if } h(x) = x \in \Pi, \\ y & \text{if either } h(x) \neq x \in \Pi \text{ or } a \neq x \in C \backslash \Pi, \\ a & \text{if } x = a, \end{cases}$$

where $y \neq x$ is the largest point in C such that $y \leq_\ell x$. One can verify that $f(x)$ is an increasing mapping in terms of the lexicographic ordering from C to itself with $f(a) = a$ and $f(b) \neq b$. Therefore, **the Tarski's fixed point problem under the componentwise ordering can be reduced to the class of discrete fixed point problems.**

The well-known equal-sums problem in Papadimitriou (1994) can be stated as follows: Given n positive integers, ν_i, $i = 1, 2, \ldots, n$, with $\sum_{i=1}^{n} \nu_i < 2^n - 1$, find two distinct subsets of these integers with the same sum. Since any n positive integers can form $2^n - 1$ nonempty subsets, hence, there always exist two distinct subsets with the same sum. It is easy to see that the equal-sums problem is equivalent to the problem of finding a nonzero solution of

$$\sum_{i=1}^{n} \nu_i x_i = 0,\ x_i \in \{-1, 0, 1\},\ i = 1, 2, \ldots, n. \tag{1}$$

Let $C = \{x \in \mathbb{Z}^n \mid a \leq x \leq b\}$, where $a = (-1, -1, \ldots, -1) \in \mathbb{Z}^n$ and $b = (1, 1, \ldots, 1) \in \mathbb{Z}^n$. For $x \in C$, we define

$$f(x) = \begin{cases} y & \text{if } \sum_{i=1}^{n} \nu_i x_i \neq 0 \text{ and } x \neq a, \\ x & \text{if either } x = a \text{ or } \sum_{i=1}^{n} \nu_i x_i = 0 \text{ and } x \neq 0, \\ (0, 0, \ldots, 0, -1) \in \mathbb{Z}^n & \text{if } x = 0, \end{cases}$$

where $y \neq x$ is the largest point in C such that $y \leq_\ell x$. Clearly, $f(x)$ is an increasing mapping in terms of the lexicographic ordering from C to itself with $f(a) = a$ and $f(b) \neq b$. Moreover, every fixed point of $f(x)$ in C other than a is a nonzero solution of the equation (1) and every nonzero solution of the equation (1) is a fixed point of $f(x)$ in C. Hence, **the equal-sums problem can be reduced to the class of discrete fixed point problems.**

As a subclass of total search problems, the PPA (Polynomial Parity Arguments on undirected graphs) class was proposed in Papadimitriou (1994). When graphs are required to be directed, PPA becomes PPAD (Polynomial Parity Arguments on directed graphs). It was shown in Papadimitriou (1994) that the class PPAD contains the classical Brouwer and Sperner lemma fixed point problems. In the last two decades, tremendous efforts have been devoted to the class PPAD in the literature, which are referred to Chen et al (2009), Daskalakis et al (2009) and Papadimitriou (2007) and the references therein. In this paper, we show that the class of discrete fixed point problems belongs to the class of Bipartie PPAD. The basic idea of our proof is stimulated from the work on integer programming in Dang and Ye (2015).

The rest of this paper is organized as follows. A reduction of the class of discrete fixed point problems to the class of Bipartie PPAD is attained in Section 2. The reduction of equilibria of a bimatrix game to the class of discrete fixed point problems is presented in Section 3.

2 A Reduction to the Class of Bipartie PPAD

We show in the following how to reduce the problem of computing a fixed point of $f(x)$ in C other than a to the class of Bipartie PPAD in polynomial time. Graph Φ is defined as follows. Nodes of Φ consist of all integer points in C and every integer point $x \in C$ with $f(x) = x$ and $x \neq a$ contributes two nodes x^E and x^S of graph Φ. It is a convention that $x^E \leq_\ell x^S$. There is a directed edge from node x to node y of graph Φ if $x \leq_\ell y$, $\{w \in C \mid x \leq_\ell w \leq_\ell y\} = \emptyset$ and $x \neq y$. There is no edge between node x^E and node x^S if $f(x) = x$ and $x \neq a$.

The degree of each node in graph Φ is determined as follows.

1. Consider node x with $f(x) \neq x$, $x \neq a$ and $x \neq b$. Let $y \neq x$ be the largest point in C such that $y \leq_\ell x$ and $w \neq x$ the smallest point in C such that $x \leq_\ell w$.

   (a) If $f(w) \neq w$ and either $f(y) \neq y$ or $y = a$, then node x is adjacent to the pair of node y and node w: $y \rightarrow x \rightarrow w$.

   (b) If $f(w) \neq w$ and $f(y) = y \neq a$, then node x is adjacent to the pair of node y^S and node w: $y^S \rightarrow x \rightarrow w$.

   (c) If $f(w) = w$ and either $f(y) \neq y$ or $y = a$, then node x is adjacent to the pair of node y and node w^E: $y \rightarrow x \rightarrow w^E$.

   (d) If $f(w) = w$ and $f(y) = y \neq a$, then node x is adjacent to the pair of node y^S and w^E: $y^S \rightarrow x \rightarrow w^E$.

   Therefore, node x has degree two (a balanced node).

2. Consider node a. Let $w \neq a$ be the smallest point in C such that $a \leq_\ell w$.

   (a) If $f(w) = w$, then node a is only adjacent to node w^E: $a \rightarrow w^E$.

   (b) If $f(w) \neq w$, then node a is only adjacent to node w: $a \rightarrow w$.

   Therefore, node a has degree one (an unbalanced node).

3. Consider node b. Let $y \neq b$ be the largest point in C such that $y \leq_\ell b$.

   (a) If $f(y) = y$, then node b is only adjacent to node y^S: $y^S \to b$.

   (b) If $f(y) \neq y$, then node b is only adjacent to node y: $y \to b$.

   Therefore, node b has degree one (an unbalanced node).

4. Consider node x^E. Let $y \neq x$ be the largest point in C such that $y \leq_\ell x$.

   (a) If $f(y) = y \neq a$, then node x^E is only adjacent to node y^S: $y^S \to x^E$.

   (b) If either $f(y) \neq y$ or $y = a$, then node x^E is only adjacent to node y: $y \to x^E$.

   Therefore, node x^E has degree one (an unbalanced node).

5. Consider node x^S. Let $w \neq x$ be the smallest point in C such that $x \leq_\ell w$.

   (a) If $f(w) = w$, then node x^S is only adjacent to node w^E: $x^S \to w^E$.

   (b) If $f(w) \neq w$, then node x^S is only adjacent to node w: $x^S \to w$.

   Therefore, node x^S has degree one (an unbalanced node).

From the construction of graph Φ, one can see that each node of Φ belongs uniquely to one of these five categories. The above results show that the degree of each node of Φ is either one or two and that only nodes in the set $\{a, b\} \cup \{x^E, x^S \mid f(x) = x \neq a,\ x \in C\}$ have degree one. Since $\leq_\ell$ yields a complete order on C, hence, **each connected component of graph Φ is a finite simple path, in which each of both end nodes has degree one and is given by one of nodes in $\{a, b\} \cup \{x^E, x^S \mid f(x) = x \neq a,\ x \in C\}$.**

Example 1. *Consider a special equal-sums problem given by*

$$x_1 + 2x_2 + 3x_3 = 0,\ x = (x_1, x_2, x_3) \in \{-1, 0, 1\}^3,$$

which has two nonzero solutions: $(-1,-1,1)$ *and* $(1,1,-1)$*. Regarding this problem, we have* $C = \{x \in \mathbb{Z}^3 \mid a \leq x \leq b\}$ *with* $a = (-1,-1,-1)$ *and* $b = (1,1,1)$*. Graph* Φ *for this problem consists of three paths:*

Path 1 : $(-1,-1,-1) \rightarrow (-1,-1,0) \rightarrow (-1,-1,1)^E$;

Path 2 : $(-1,-1,1)^S \rightarrow (-1,0,-1) \rightarrow (-1,0,0) \rightarrow (-1,0,1) \rightarrow (-1,1,-1)$ $\rightarrow (-1,1,0) \rightarrow (-1,1,1) \rightarrow (0,-1,-1) \rightarrow (0,-1,0) \rightarrow (0,-1,1) \rightarrow (0,0,-1)$ $\rightarrow (0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,-1) \rightarrow (0,1,0) \rightarrow (0,1,1) \rightarrow (1,-1,-1)$ $\rightarrow (1,-1,0) \rightarrow (1,-1,1) \rightarrow (1,0,-1) \rightarrow (1,0,0) \rightarrow (1,0,1) \rightarrow (1,1,-1)^E$;

Path 3 : $(1,1,-1)^S \rightarrow (1,1,0) \rightarrow (1,1,1)$.

Clearly, nodes a and b are the only known end nodes of graph Φ and all other end nodes of graph Φ are unbalanced nodes satisfying that each of them yields a fixed point of f. Let x be a fixed point f with $x \neq a$. Then we have $a \leq_\ell x \leq_\ell b$. Thus, nodes a and b are end nodes of two different paths. These results together with the definition of a PPAD graph in Papadimitriou (1994) yield the following conclusion.

Theorem 1. Φ *is a Bipartie PPAD graph.*

Proof.

1. Clearly, Φ is a directed graph defined on a finite but exponentially large set of vertices.

2. From the construction of Φ, one can see that each node has indegree and outdegree at most one.

3. (a). Any x with $x \in C$ is a node of Φ if either $f(x) \neq x$ or $x = a$. Any x with $x \in C$ contributes two nodes x^E and x^S if $f(x) = x \neq a$. (b). For any given node of Φ, one can obtain in polynomial time its neighbors (one or two of them).

4. There are exactly two known nodes as end nodes of two different paths, which are given by node a and node b.

5. Any end node of the graph other than node a and node b are solutions of the problem. This completes the proof. $\square$

From Example 1, one can see that graph Φ for the problem in Example 1 is indeed a Bipartie PPAD graph. As a corollary of Theorem 1, we come to the main result of this paper.

Corollary 1. *Computing a fixed point of f in C other than a is in the class of Bipartie PPAD.*

3 Reduction of Equilibria of a Bimatrix Game to the Class of Discrete Fixed Point Problems

As that in Dang and Ye (2018), consider a bimatrix game with rational payoff matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$. We assume, without loss of generality, that A and B are positive. Let

$$D = \begin{pmatrix} \mathbf{0} & -A \\ -B^{\top} & \mathbf{0} \end{pmatrix}$$

and e be a vector of ones in $\mathbb{R}^{m+n}$. The linear complementarity problem (LCP) corresponding to the bimatrix game is given by

$$(LCP): \quad Dx + e \geq \mathbf{0}, \; x \geq \mathbf{0}, \; x^{\top}(Dx + e) = 0.$$

It is well known that every nonzero solution of the LCP yields a Nash equilibrium of the bimatrix game and every Nash equilibrium of the bimatrix game yields a nonzero solution of the LCP. Let M be a sufficiently large positive integer and $d_i^{\top}$ denote the ith row of D for $i = 1, 2, \ldots, m+n$. Then an equivalent mixed-integer programming formulation to the LCP of a bimatrix game is given by

$$\begin{aligned}
&-Dx \le e,\ -x \le \mathbf{0},\\
&-Mz_i + x_i \le 0,\ i = 1, 2, \ldots, m+n,\\
&Mz_i + d_i^\top x \le M - 1,\ i = 1, 2, \ldots, m+n,\\
&-z \le \mathbf{0},\ z \le e,\ -\sum_{i=1}^{m+n} z_i \le -1,
\end{aligned}$$

where z_i, $i = 1, 2, \ldots, m+n$, are integer variables and x_i, $i = 1, 2, \ldots, m+n$, are continuous variables. Let

$$G = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ -MI \\ MI \\ -I \\ I \\ -e^\top \end{pmatrix},\ F = \begin{pmatrix} -D \\ -I \\ I \\ D \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \text{ and } b = \begin{pmatrix} e \\ \mathbf{0} \\ \mathbf{0} \\ (M-1)e \\ \mathbf{0} \\ e \\ -1 \end{pmatrix},$$

where I is an $(m+n) \times (m+n)$ identity matrix. With these notations, the mixed-integer programming problem can be rewritten as

$$\begin{aligned}
\max\quad & \mathbf{0}^\top z + \mathbf{0}^\top x = 0\\
\text{s.t.}\quad & Gz + Fx \le b,\ z_i, i = 1, 2, \ldots, m+n, \text{are integers.}
\end{aligned}$$

An application of Benders' decomposition yields

$$\begin{aligned}
\max\quad & \mathbf{0}^\top z + \lambda = 0\\
\text{s.t.}\quad & u^\top Gz + \lambda \le u^\top b \text{ for all } u \ge \mathbf{0} \text{ with } u^\top F = 0,\\
& z_i,\ i = 1, 2, \ldots, m+n,\ \text{integers.}
\end{aligned}$$

Thus, finding an Nash equilibrium of a bimatrix game is equivalent to the problem of finding an integer point z such that

$$u^\top Gz \le u^\top b \text{ for all } u \ge \mathbf{0} \text{ with } u^\top F = 0.$$

Let $P_0 = \{u \in \mathbb{R}^{6(m+n)+1} \mid F^\top u = 0 \text{ and } u \ge \mathbf{0}\}$. Since P_0 is a polyhedral cone, there is a finite number of rational vectors $u^i \in \mathbb{R}^{6(m+n)+1}$,

$i = 1, 2, \ldots, p_0$, such that $P_0 = \text{cone}\{u^i,\ i = 1, 2, \ldots, p_0\}$. Rewriting u as $(\widetilde{u}^1, \widetilde{u}^2, \widetilde{u}^3, \widetilde{u}^4, \widetilde{u}^5)$ with $\widetilde{u}^i \in \mathbb{R}^{m+n}$, $i = 1, 2, 3, 4$, and $\widetilde{u}^5 \in \mathbb{R}^{2(m+n)+1}$, we have $F^\top u = -D^\top \widetilde{u}^1 + D^\top \widetilde{u}^4 - \widetilde{u}^2 + \widetilde{u}^3$. Let $v = \widetilde{u}^4 - \widetilde{u}^1$. It follows that $\{\widetilde{u}^3 \mid -D^\top \widetilde{u}^1 + D^\top \widetilde{u}^4 - \widetilde{u}^2 + \widetilde{u}^3 = 0,\ \widetilde{u}^i \geq \mathbf{0},\ i = 1, 2, 3, 4\}$ is equal to $\{\widetilde{u}^3 \mid D^\top v + \widetilde{u}^3 \geq \mathbf{0},\ \widetilde{u}^3 \geq \mathbf{0}\}$. Let $C_1 = \{(v, w) \in \mathbb{R}^{2(m+n)} \mid D^\top v + w \geq \mathbf{0},\ w \geq \mathbf{0}\}$. Since C_1 is a polyhedral cone and D is an $(m+n) \times (m+n)$ matrix, there exist rational vectors (v^i, w^i), $i = 1, 2, \ldots, p_1$, with $p_1 \leq 4(m+n)^2$ such that $C_1 = \text{cone}\{(v^i, w^i), i = 1, 2, \ldots, p_1\}$. Note that, for each v^i, the polyhedron $\{\widetilde{u}^1 \mid \widetilde{u}^1 + v^i \geq \mathbf{0},\ \widetilde{u}^1 \geq \mathbf{0}\}$ has exactly one rational vertex and $m + n$ rational extreme rays. These results together with $\widetilde{u}^5 \in \mathbb{R}^{2(m+n)+1}$ imply that p_0 is bounded by $4(2(m+n)+1)(m+n+1)(m+n)^2$. Therefore, finding a Nash equilibrium of a bimatrix game is equivalent to the problem of finding an integer point z such that $u^{i\top} Gz \leq u^{i\top} b,\ i = 1, 2, \ldots, p_0$, with $p_0 \leq 4(2(m+n)+1)(m+n+1)(m+n)^2$. The above reduction can also be found in Dang and Ye (2018).

Let $P = \{x \in \mathbb{R}^{m+n} \mid u^{i\top} Gx \leq u^{i\top} b,\ i = 1, 2, \ldots, p_0\}$. Solving linear programs $x_i^{\max} = \max\limits_{x \in P} x_i$ and $x_i^{\min} = \min\limits_{x \in P} x_i$ for $i = 1, 2, \ldots, m+n$, we obtain $x^{\max} = (x_1^{\max}, x_2^{\max}, \ldots, x_{m+n}^{\max})^\top$ and $x^{\min} = (x_1^{\min}, x_2^{\min}, \ldots, x_{m+n}^{\min})^\top$. Let $x^L = \lfloor x^{\min} \rfloor - e = (\lfloor x_1^{\min} \rfloor - 1, \lfloor x_2^{\min} \rfloor - 1, \ldots, \lfloor x_{m+n}^{\min} \rfloor - 1)^\top$ and $x^U = \lceil x^{\max} \rceil + e = (\lceil x_1^{\max} \rceil + 1, \lceil x_2^{\max} \rceil + 1, \ldots, \lceil x_{m+n}^{\max} \rceil + 1)^\top$, where e is a vector of ones in $\mathbb{R}^{m+n}$. Let $C = \{x \in \mathbb{Z}^{m+n} \mid a \leq x \leq b\}$ with $a = x^L$ and $b = x^U$. Then $z \in C$ if z is an integer point in P. Clearly $a \notin P$ and $b \notin P$. For $x \in C$, we define

$$f(x) = \begin{cases} x & \text{if either } x \in P \text{ or } x = a, \\ y & \text{if } x \notin P \text{ and } x \neq a, \end{cases}$$

where $y \neq x$ is the largest point in C such that $y \leq_\ell x$. One can easily show that $f(x)$ is an increasing mapping in terms of the lexicographic ordering from C to itself with $f(a) = a$ and $f(b) \neq b$. Therefore the problem of equilibria of a bimatrix game can be reduced to the class of discrete fixed point problems.

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