LECTURE II

METHOD OF STEEPEST DESCENT

> He (Debye) found a very powerful technique for evaluating some integrals $\cdots$. (This invention) required only a clear picture of the relevant feature of the mathematical object (*not its abstraction*) and a willingness to discard that which was not essential to the task [my italics].
>
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1. Introduction

By name, this is probably the best known method for finding the asymptotic behavior of integrals of the form

$$I(\lambda) = \int_C g(z)e^{\lambda f(z)}dz, \tag{1.1}$$

where $f(z)$ and $g(z)$ are analytic functions, λ is a large parameter and C is a contour in the z-plane. However, it is probably also the one which is least understood by non-specialists. Excellent survey-type articles have been written on this subject; here we mention only the following two: Wyman [17] and Olver [8].

Debye devised this method in 1909 in order to derive asymptotic expansions of Bessel functions of large order. His basic idea is to deform the contour C into a new path of integration C' so that the following conditions hold:

(i) C' passes through one or more zeros of $f'(z)$;

(ii) the imaginary part of $f(z)$ is constant on C'.

To obtain a geometric interpretation of the new path of integration, we write

$$f(z) = u(x, y) + iv(x, y),$$

where $z = x+iy$ and u and v are real. If u is treated as a third axis orthogonal to both x and y, then the equation $u = u(x, y)$ defines a surface S in the (x, y, u) space. Suppose that $z_0 = x_0 + iy_0$ is a zero of $f'(z)$. Then, by the Cauchy-Riemann equation, $f'(z) = u_x - iu_y$. Thus, $f'(z_0) = 0$ implies

$$u_x(x_0, y_0) = u_y(x_0, y_0) = 0,$$

i.e., (x_0, y_0) is a critical point of $u(x, y)$. Since u is a harmonic function, $u(x, y)$ cannot have a maximum or a minimum at an interior point. Therefore, (x_0, y_0) must be a saddle point of $u(x, y)$. For this reason, we call z_0 a *saddle point* of $f(z)$.

The shape of the surface S on the (x, y) plane can be represented by drawing the level curves on which u is constant. The curves $v =$ constant are the orthogonal trajectories of the level curves, but so are the projections of the paths of steepest ascent or descent on the surface. The term *steepest descent* stems from condition (ii) above.

We suppose that it is possible to deform the original path of integration into a steepest path $v(x, y) =$ constant $=$ Im $f(z_0)$. On this path, we have

$$f(z) = f(z_0) - \tau, \tag{1.2}$$

where τ is real and is either monotonically increasing or monotonically decreasing. The integrand in (1.1) becomes $g(z)e^{\lambda f(z_0)-\lambda\tau}$. On a path where $\tau \to -\infty$, the integral may be divergent. For this reason we choose paths on which τ is positive and increasing. These are the paths of steepest descent from saddle points. We suppose that the original path of integration C in (1.1) can be deformed into an equivalent path consisting of paths of steepest descent through a saddle point. Then our problem is reduced to finding the asymptotic behavior of integrals of the form

$$e^{\lambda f(z_0)} \int_0^\infty g(z) \frac{dz}{dt} e^{-\lambda\tau} d\tau. \tag{1.3}$$

The above brief explanation of the method of steepest descent will be made clearer by the example of Airy's integral given in Sec. 2. The results for Airy's integral will be used to illustrate the Stokes phenomenon in Sec. 3. In Sec. 4, we present a modified version of the steepest descent method introduced by Berry and Howls [5] in 1991. The final two sections contain a brief introduction to a sub-area of asymptotics which is now known as *exponential asymptotics*.

2. The Airy Integral

This integral is defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_L \exp\left(\frac{1}{3}t^3 - zt\right) dt, \tag{2.1}$$

where L is any contour which begins at infinity in the sector $-\frac{1}{2}\pi < \arg t < -\frac{1}{6}\pi$ and ends at infinity in the sector $\frac{1}{6}\pi < \arg t < \frac{1}{2}\pi$. To transform it into the form given in (1.1), we first assume that z is real and positive, and then make the change of variable $t = z^{1/2}u$. This gives

$$\mathrm{Ai}(z) = \frac{z^{1/2}}{2\pi i} \int_L \exp\left\{ z^{3/2}\left(\frac{1}{3}u^3 - u\right)\right\} du.$$

Once this identity has been established, the restriction on z can be removed by using analytic continuation. Put

$$\xi = \frac{2}{3}z^{3/2} \qquad \text{and} \qquad f(u) = \frac{1}{2}(u^3 - 3u) \tag{2.2}$$

so that

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\left(\frac{3}{2}\xi\right)^{1/3} \int_L e^{\xi f(u)} du. \tag{2.3}$$

The saddle points of $f(u)$ are at $u = \pm 1$. Clearly, $f(\pm 1) = \mp 1$. If we write $u = s + it$, then we have

$$f(u) = \frac{1}{2}[(s^3 - 3st^2 - 3s) + i(3s^2 t - t^3 - 3t)].$$

Since $\mathrm{Im}\, f(\pm 1) = 0$, $\mathrm{Im}\, f = 0$ implies either $t = 0$ or $t^2 - 3s^2 + 3 = 0$. On $t = 0$, we have $\mathrm{Re}\ f = \frac{1}{2}(s^3 - 3s)$, which has a local minimum at $s = 1$ and a local maximum at $s = -1$. Thus, near $s = 1, t = 0$ is a steepest ascent curve. The other equation $t^2 - 3s^2 + 3 = 0$ represents two branches of a hyperbola with the asymptotes $t = \pm\sqrt{3}s$. On this hyperbola, $\mathrm{Re}\, f = -4s^3 + 3s$, which decreases for $s > 1$. From Figure 1, it is clear that the branch of the hyperbola on the right half plane is our desired path of steepest descent through the saddle point $u = 1$. (In Figure 1, arrows indicate the direction in which $\mathrm{Re}\ f$ decreases.)

Deforming the original contour L into this path of steepest descent, we can write

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\left(\frac{3}{2}\xi\right)^{1/3} \left(\int_1^{\infty e^{i\pi/3}} - \int_1^{\infty e^{-i\pi/3}}\right) e^{\xi f(u)} du. \tag{2.4}$$

In both integrals above, $f(u) - f(1)$ is real and has a maximum at $u = 1$. Also, $f(u) - f(1)$ is decreasing as u moves away from $u = 1$. Set

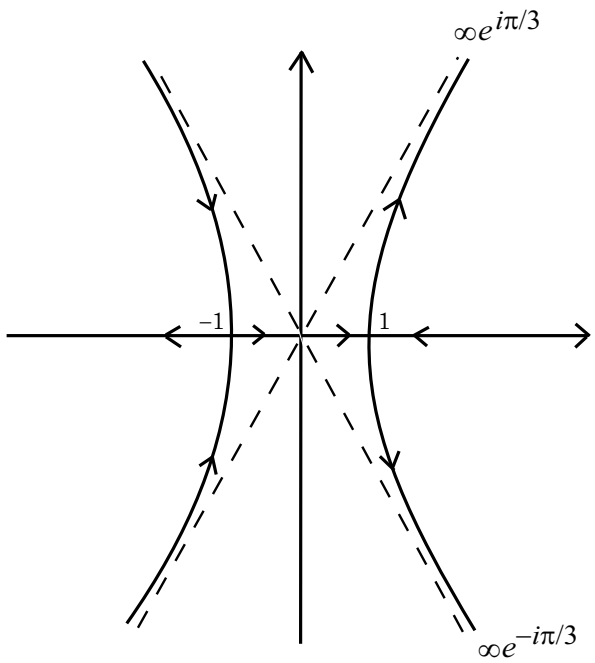

Figure 1. Steepest paths for $\mathrm{Ai}(z)$

$$\tau = f(1) - f(u) = -1 + \frac{3}{2}u - \frac{1}{2}u^3 = -\frac{3}{2}(u-1)^2 - \frac{1}{2}(u-1)^3. \tag{2.5}$$

On our steepest descent path, τ is real; cf. Eq. (1.2). From (2.5), we have

$$\pm i\left(\frac{2}{3}\tau\right)^{1/2} = (u-1)\left[1 + \frac{1}{3}(u-1)\right]^{1/2},$$

where $[\cdots]^{1/2}$ is that branch which reduces to 1 at $u = 1$. By Lagrange's formula for the reversion of series

$$u^{\pm} = 1 + \sum_{n=1}^{\infty} a_n \left[\pm i\left(\frac{2}{3}\tau\right)^{1/2}\right]^n,$$

where

$$a_n = \frac{1}{n!}\frac{d^{n-1}}{du^{n-1}}\left[1 + \frac{1}{3}(u-1)\right]^{-n/2}\Bigg|_{u=1}.$$

Since u^+ enters the first quadrant for increasing τ, we must take u^+ for the first integral in (2.4). Similarly, we take u^- for the second integral there. Equation (2.4) then becomes

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\left(\frac{3}{2}\xi\right)^{1/3} e^{-\xi} \int_0^{\infty} \left(\frac{du^+}{d\tau} - \frac{du^-}{d\tau}\right) e^{-\xi\tau} d\tau. \tag{2.6}$$

Since na_n is the coefficient of $(u-1)^{n-1}$ in the Taylor expansion of $\left[1 + \frac{1}{3}(u-1)\right]^{-n/2}$ at $u = 1$, it is easily shown that

$$a_n = \frac{(-1)^{n-1}\Gamma\left(\frac{3}{2}n - 1\right)}{n!\,\Gamma\left(\frac{1}{2}n\right)3^{n-1}}. \tag{2.7}$$

From this, it follows that

$$\frac{du^+}{d\tau} - \frac{du^-}{d\tau} = 2i \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(3m + \frac{1}{2}\right)\left(m + \frac{1}{2}\right)}{(2m+1)!\,\Gamma\left(m + \frac{1}{2}\right)9^m} \left(\frac{2}{3}\right)^{m+\frac{1}{2}} \tau^{m-\frac{1}{2}}.$$

Termwise integration gives

$$\mathrm{Ai}(z) \sim \frac{1}{2\pi z^{1/4}} \exp\left(-\frac{2}{3}z^{3/2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(3m + \frac{1}{2}\right)}{(2m)!\,9^m} z^{-3m/2} \tag{2.8}$$

as $z \to \infty$ in $|\arg z| < \frac{1}{3}\pi$.

The sector of validity of (2.8) can be extended to $|\arg z| < \pi$ by rotating the path of integration in (2.6). This way of extending the region of validity

is frequently used in asymptotics; see, e.g., [12, p.26]. However, the result in (2.8) can not be valid in a wider sector, since $\mathrm{Ai}(z)$ is a single-valued function while the factor multiplying the infinite series in (2.8) is not. In fact, from (2.1), it can be shown that $\mathrm{Ai}(z)$ has the Maclaurin expansion

$$\mathrm{Ai}(z) = 3^{-2/3}\left[\sum_{n=0}^{\infty} \frac{z^{3n}}{3^{2n} n!\,\Gamma\left(n+\frac{2}{3}\right)} - 3^{-2/3}\sum_{n=0}^{\infty} \frac{z^{3n+1}}{3^{2n} n!\,\Gamma\left(n+\frac{4}{3}\right)}\right]; \tag{2.9}$$

that is, $\mathrm{Ai}(z)$ is an entire function. The material in this section is taken from [12, Chap. II, Sec. 4], where more examples can be found on the method of steepest descent.

3. Stokes' Phenomenon

From (2.1), it can be shown that $\mathrm{Ai}(z)$ is a solution of

$$\frac{d^2 w}{dz^2} - zw = 0. \tag{3.1}$$

Clearly, $\mathrm{Ai}(wz)$ and $\mathrm{Ai}(w^2 z)$ are also solutions of this equation, where $w = e^{2\pi i/3}$. Using Cauchy's theorem, it can also be shown that the three solutions are connected by the relation

$$\mathrm{Ai}(z) + w\mathrm{Ai}(wz) + w^2\mathrm{Ai}(w^2 z) = 0. \tag{3.2}$$

If $\pi/3 < \arg z < 5\pi/3$, then $-\pi/3 < \arg(w^{-1}z) < \pi$ and $-\pi < \arg(w^{-2}z) < \pi/3$. From (2.8) and (3.2), it follows that

$$\begin{aligned}\mathrm{Ai}(z) \sim & \frac{1}{2\pi z^{1/4}} \exp\left(-\frac{2}{3}z^{3/2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(3m+\frac{1}{2}\right)}{(2m)!\,9^m} z^{-3m/2} \\ & + \frac{i}{2\pi z^{1/4}} \exp\left(\frac{2}{3}z^{3/2}\right) \sum_{m=0}^{\infty} \frac{\Gamma\left(3m+\frac{1}{2}\right)}{(2m)!\,9^m} z^{-3m/2}\end{aligned} \tag{3.3}$$

as $z \to \infty$ in $\pi/3 < \arg z < 5\pi/3$. If we let $u_{\pm}(z)$ denote the formal series

$$u_{\pm}(z) = \frac{1}{2\pi z^{1/4}} \exp\left(\pm\frac{2}{3}z^{3/2}\right) \sum_{m=0}^{\infty} \frac{(\pm 1)^m \Gamma\left(3m+\frac{1}{2}\right)}{(2m)!\,9^m} z^{-3m/2}, \tag{3.4}$$

then (2.8) and (3.3) can be written, respectively, as

$$\mathrm{Ai}(z) \sim u_-(z) \tag{3.5}$$

as $z \to \infty$ in $-\pi < \arg z < \pi$ and.

$$\mathrm{Ai}(z) \sim u_-(z) + iu_+(z) \tag{3.6}$$

as $z \to \infty$ in $\pi/3 < \arg z < 5\pi/3$.

Note that both results (3.5) and (3.6) are valid in the common sector $\pi/3 < \arg z < \pi$, and that there is no inconsistency since the first term on the right-hand side of (3.6) is *dominant* (exponentially increasing) and the second term is *recessive* (exponentially decreasing) in the sector $\pi/3 < \arg z < \pi$. However, the roles of these two terms are interchanged in the sector $-\pi/3 < \arg z < \pi/3$; hence it is mandatory to drop $u_+(z)$ from the asymptotic expansion of $\mathrm{Ai}(z)$. By introducing a constant (coefficient) C, which is 0 for $\arg z \in (-\pi/3, \pi/3)$ and i for $\arg z \in (\pi/3, 5\pi/3)$, the two results in (3.5) and (3.6) can be combined into one, namely

$$\mathrm{Ai}(z) \sim u_-(z) + Cu_+(z) \tag{3.7}$$

as $z \to \infty$ in $\arg z \in (-\pi, 5\pi/3)$. The coefficient C is called a *Stokes multiplier*, and is domain dependent. The discontinuous change of the coefficient C, when the argument of z changes in a continuous manner, is known as *Stokes' phenomenon*.

Returning to (3.4), we let

$$S_+(z) = \frac{2}{3}z^{3/2} \qquad \text{and} \qquad S_-(z) = -\frac{2}{3}z^{3/2}.$$

It can be verified that the behavior of $u_+(z)$ and $u_-(z)$ are most unequal on the curves

$$\mathrm{Im}\{S_+(z) - S_-(z)\} = 0 \tag{3.8}$$

and they are nearly equal on the curves given by

$$\mathrm{Re}\{S_+(z) - S_-(z)\} = 0. \tag{3.9}$$

The curves given in (3.8) and (3.9) are known, respectively, as the *Stokes* and *anti-Stokes lines*. In the case of the Airy function, it is easily seen that the rays $\arg z = 0, \pm 2\pi/3$ are the Stokes lines and the rays $\arg z = \pm\pi/3, \pm\pi$ are the anti-Stokes lines.

Since $\mathrm{Ai}(z)$ on the left-hand side of (3.7) is an analytic function, it is rather undesirable to have a discontinuous coefficient C on the right-hand side of the equation. In 1989, Berry [2] gave a different interpretation of the Stokes phenomenon. In his view, if one truncates the series $u_-(z)$ at an "optimal" place, then the coefficient of the series $u_+(z)$ should be a continuous function of $\arg z$, instead of a discontinuous constant. We shall illustrate Berry's theory with the simple Airy function given in (2.1). Our approach is based on a modified version of the steepest descent method introduced by Berry and Howls [4] in 1990. The material in the next two sections is taken nearly verbatim from Wong [13]. (Permission has been obtained from Kluwer Academic Publishers.)

4. Adjacent Saddle and Adjacent Contour

In (2.3), we let

$$\theta := \arg \xi, \tag{4.1}$$

and consider the steepest descent curves

$$\Gamma_{\pm 1}(\theta): \qquad \arg\{\xi[f(\pm 1) - f(u)]\} = \arg\{e^{i\theta}[f(\pm 1) - f(u)]\} = 0, \tag{4.2}$$

i.e., curves on which $\mathrm{Im}\{\xi[f(\pm 1) - f(u)]\} = 0$ and $\mathrm{Re}\{\xi[f(\pm 1) - f(u)]\} > 0$. Deforming the contour L in (2.3) into $\Gamma_1(\theta)$, we obtain

$$\mathrm{Ai}(z) = \frac{e^{-\xi}}{2\pi i}\left(\frac{3}{2}\xi\right)^{1/3} \int_{\Gamma_1(\theta)} e^{\xi[f(u)+1]} du. \tag{4.3}$$

If we introduce the notations

$$I^{(\pm 1)}(\xi) := \xi^{1/2} \int_{\Gamma_1(\theta)} e^{\xi[f(u)\pm 1]} du, \tag{4.4}$$

then (4.3) can be written as

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\left(\frac{3}{2}\right)^{1/3} \xi^{-1/6} e^{-\xi} I^{(1)}(\xi). \tag{4.5}$$

In the integral (4.4), we make the change of variable

$$-\tau = \xi[f(u) + 1]. \tag{4.6}$$

For $u \in \Gamma_1(\theta)$, τ is real and positive; cf. (4.2). As in (2.5), we now expand $f(u)$ into a Taylor series at $u = 1$. Lagrange's inversion formula again gives

$$u^{\pm} = 1 + \sum_{n=1}^{\infty} a_n \left(\pm i\sqrt{\frac{2\tau}{3\xi}}\right)^n, \tag{4.7}$$

where the coefficients a_n are given in (2.7). Note that here u is a function of τ and ξ. By breaking the integration path $\Gamma_1(\theta)$ in (4.4) at $u = 1$, we can rewrite $\mathrm{I}^{(1)}(\xi)$ as

$$\begin{aligned} \mathrm{I}^{(1)}(\xi) &= \xi^{1/2} \int_0^{\infty} \left[\frac{du^+}{d\tau} - \frac{du^-}{d\tau}\right] e^{-\tau} d\tau \\ &= \xi^{-1/2} \int_0^{\infty} \left[\frac{1}{f'(u^-(\tau))} - \frac{1}{f'(u^+(\tau))}\right] e^{-\tau} d\tau; \end{aligned} \tag{4.8}$$

cf. (2.6).

The first step in the Berry-Howls method [5] is to use Cauchy's residue theorem to represent the integrand in (4.8) as a contour integral. To see this, we let $C_1(\theta)$ be a positively oriented curve surrounding the steepest-descent path $\Gamma_1(\theta)$. Since $\Gamma_1(\theta)$ is an infinite contour, $C_1(\theta)$ actually consists of two infinite curves embracing $\Gamma_1(\theta)$; see Figure 2. We now recall the formula

$$\mathrm{Res}\left\{\frac{Q(u)}{P(u)}; u_0\right\} = \frac{Q(u_0)}{P'(u_0)},$$

where $P(u)$ and $Q(u)$ are analytic functions with $P(u_0) = 0, P'(u_0) \neq 0$ and $Q(u_0) \neq 0$. Take $P(u) \equiv \xi[-1 - f(u)] - \tau, Q(u) \equiv [-1 - f(u)]^{1/2}$ and $u_0 = u^{\pm}(\tau)$. Then

$$\frac{Q(u^{\pm}(\tau))}{P'(u^{\pm}(\tau))} = \mathrm{Res}\left\{\frac{Q(u)}{P(u)}; u^{\pm}(\tau)\right\} = \frac{1}{2\pi i}\int_{C^{\pm}} \frac{Q(u)}{P(u)} du, \tag{4.9}$$

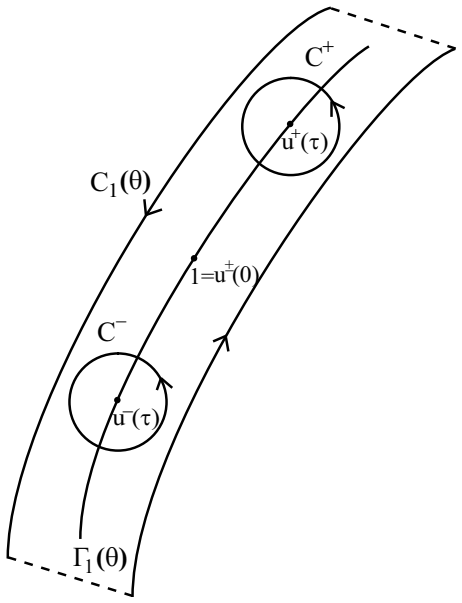

Figure 2. Contour $C_1(\theta)$.

where C^+ and C^- are the two closed contours shown in Figure 2. Since $Q(u^{\pm}(\tau)) = (\tau/\xi)^{1/2}$ and $P'(u^{\pm}(\tau)) = -\xi f'(u^{\pm}(\tau))$, it follows from (4.9) that

$$\frac{1}{f'(u^-(\tau))} - \frac{1}{f'(u^+(\tau))} = \frac{1}{2\pi i}\frac{\xi^{3/2}}{\tau^{1/2}}\int_{C_1(\theta)} \frac{[-1 - f(u)]^{1/2}}{\xi[-1 - f(u)] - \tau} du. \tag{4.10}$$

Inserting (4.10) into (4.8), we get a double integral for $\mathrm{I}^{(1)}(\xi)$. Upon interchanging the order of integration, we obtain

$$\mathrm{I}^{(1)}(\xi) = \frac{1}{2\pi i}\int_{C_1(\theta)} [-1 - f(u)]^{-1/2} \int_0^{\infty} \frac{e^{-\tau}\tau^{-1/2}}{1 - \{\tau/\xi[-1 - f(u)]\}} d\tau du. \tag{4.11}$$

The geometric series

$$\frac{1}{1-x} = \sum_{s=0}^{N-1} x^s + \frac{x^N}{1-x}$$

then gives the asymptotic expansion

$$\mathrm{I}^{(1)}(\xi) = \sum_{s=0}^{N-1} c_s \xi^{-s} + R_N(\xi), \tag{4.12}$$

where

$$c_s = \frac{\Gamma\left(s+\frac{1}{2}\right)}{2\pi i}\int_{C_1(\theta)}[-1-f(u)]^{-s-\frac{1}{2}}du \tag{4.13}$$

and

$$\begin{aligned} R_N(\xi) = \frac{\xi^{-N}}{2\pi i}\int_0^\infty e^{-\tau}\tau^{N-\frac{1}{2}}\int_{C_1(\theta)}&[-1-f(u)]^{-N-\frac{1}{2}} \\ &\times\frac{1}{1-\{\tau/\xi[-1-f(u)]\}}dud\tau. \end{aligned} \tag{4.14}$$

The coefficients c_s can be evaluated exactly, and we have

$$c_s = i\frac{(-1)^s\Gamma\left(3s+\frac{1}{2}\right)}{(2s)!\,9^s}\left(\frac{3}{2}\right)^{-s-\frac{1}{2}}. \tag{4.15}$$

The second important step in the Berry-Howls method is to consider all steepest descent paths $\Gamma_1(\theta)$ passing through $u=1$ for different values of θ; see Figure 3. Since $f(1)-f(-1)=-2$, the path

$$\Gamma_1(\pi): \qquad \arg\{e^{i\pi}[f(1)-f(u)]\}=0 \tag{4.16}$$

runs into the saddle point $u=-1$. Berry and Howls called $u=-1$ an *adjacent saddle* of $u=1$, and the steepest-descent path

$$\Gamma_{-1}(\pi): \qquad \arg\{e^{i\pi}[f(-1)-f(u)]\}=0 \tag{4.17}$$

an *adjacent contour*. Deforming the contour $C_1(\theta)$ in (4.14) into $\Gamma_{-1}(\pi)$, we obtain

$$\begin{aligned} R_N(\xi) = \frac{\xi^{-N}}{2\pi i}\int_0^\infty e^{-\tau}\tau^{N-\frac{1}{2}}\int_{\Gamma_{-1}(\pi)}&[-1-f(u)]^{-N-\frac{1}{2}} \\ &\times\frac{1}{1-\{\tau/\xi[-1-f(u)]\}}dud\tau. \end{aligned} \tag{4.18}$$

In the last equation, we make the change of variable

$$\tau = t\frac{f(1)-f(u)}{f(1)-f(-1)}; \tag{4.19}$$

recall that $f(1)=-1$. Since

$$\frac{f(1)-f(u)}{f(1)-f(-1)} = 1+\frac{f(-1)-f(u)}{f(1)-f(-1)}, \tag{4.20}$$

and since the quotient on the right-hand side is real and positive when $u\in\Gamma_{-1}(\pi)$, the quotient on the left-hand side is also real and positive for $u\in$

$\Gamma_{-1}(\pi)$. Dingle [7] called the quantity $f(1) - f(-1)$ a *singulant*; see also [4]. Substituting (4.19) in (4.18), and making use of (4.20), we obtain

$$R_N(\xi) = \frac{\xi^{-N}}{2\pi i}(-2)^{-N-\frac{1}{2}} \int_0^\infty e^{-t} t^{N-\frac{1}{2}} \left(1 + \frac{t}{2\xi}\right)^{-1} \times \int_{\Gamma_{-1}(\pi)} \exp\left\{-t\frac{f(-1) - f(u)}{f(1) - f(-1)}\right\} du dt. \tag{4.21}$$

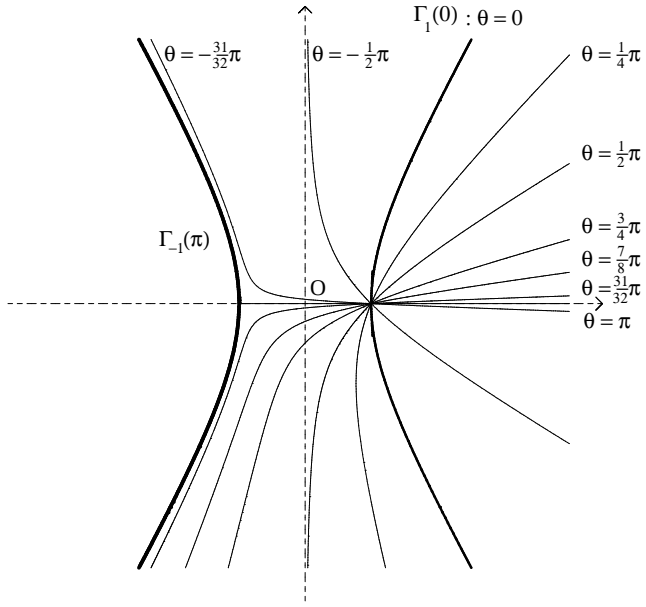

Figure 3. Contours $\Gamma_1(\theta), -\pi < \theta < \pi$

In the inner integral, we write $u = -w$. Since $f(u)$ is an odd function and $f(1) - f(-1) = -2$, it follows that

$$\int_{\Gamma_{-1}(\pi)} \exp\left\{-t\frac{f(-1) - f(u)}{f(1) - f(-1)}\right\} du = -\int_{\Gamma_1(0)} \exp\left\{\frac{t}{2}[f(w) - f(1)]\right\} dw.$$

The last integral can be expressed in terms of the integral $I^{(1)}(\xi)$ given in (4.4). Indeed, we have

$$\int_{\Gamma_{-1}(\pi)} \exp\left\{-t\frac{f(-1) - f(u)}{f(1) - f(-1)}\right\} du = -\left(\frac{t}{2}\right)^{-1/2} I^{(1)}\left(\frac{t}{2}\right). \tag{4.22}$$

Inserting (4.22) into (4.21) gives

$$R_N(\xi) = \frac{1}{2\pi}(-2\xi)^{-N} \int_0^\infty e^{-t} t^{N-1} \left(1 + \frac{t}{2\xi}\right)^{-1} I^{(1)}\left(\frac{t}{2}\right) dt. \tag{4.23}$$

Equations (4.12) and (4.23) coupled together is known as a *resurgence formula*, since the integral $\mathrm{I}^{(1)}(\xi)$ on the left-hand side of (4.12) appears again in the remainder term $R_{\mathrm{N}}(\xi)$ given in (4.23).

5. Exponential Asymptotics

To estimate the remainder term in (4.12), we first use the double integral representation of $R_N(\xi)$ given in (4.18). For convenience, we introduce the function

$$\csc_1(\varphi) = \begin{cases} |\csc\varphi| & \text{if } 0 < |\varphi| < \dfrac{\pi}{2}, \\ 1 & \text{if } \dfrac{\pi}{2} \le |\varphi| \le \pi. \end{cases} \tag{5.1}$$

It can be readily verified that if $\zeta = |\zeta| e^{i\varphi}$ then

$$\left| \frac{1}{1-\zeta} \right| \le \csc_1(\varphi). \tag{5.2}$$

We extend $\csc_1(\varphi)$ into a 2π-periodic function by defining

$$\csc_1(\varphi + 2\pi) = \csc_1(\varphi). \tag{5.3}$$

Since this is an even function, it is unbounded at $0, \pm 2\pi, \pm 4\pi, \cdots$. Take $\zeta = \tau/\xi[-1 - f(u)]$, and note that $\arg\zeta = -\pi - \theta$. By (5.2), we have

$$\left| 1 - \frac{\tau}{\xi[-1 - f(u)]} \right|^{-1} \le \csc_1(\pi + \theta) \tag{5.4}$$

for $u \in \Gamma_{-1}(\pi)$. A direct application of (5.4) to (4.18) yields

$$|R_N(\xi)| \le \frac{A_N}{|\xi|^N} \csc_1(\theta + \pi), \tag{5.5}$$

where A_N is a constant given by

$$A_N = \frac{\Gamma\left(N + \frac{1}{2}\right)}{2\pi} \int_{\Gamma_{-1}(\pi)} \left| [-1 - f(u)]^{-N-\frac{1}{2}} du \right|. \tag{5.6}$$

There is another error estimate which is probably easier to compute. We first note that $I^{(1)}(\xi) = R_0(\xi)$; see (4.12). Hence, by (5.5),

$$\left| I^{(1)}\left(\frac{t}{2}\right) \right| \le A_0 \csc_1(\pi). \tag{5.7}$$

Next, we return to (4.23). With $\zeta = -t/2\xi$, we have from (5.2)

$$|R_N(\xi)| \le \frac{B_N}{|\xi|^N} \csc_1(\pi - \theta), \tag{5.8}$$

where

$$B_N = \frac{A_0\Gamma(N)}{2^{N+1}\pi}. \tag{5.9}$$

Since the function $\csc_1(\varphi)$ is unbounded at $\varphi = 0, \pm 2\pi, \pm 4\pi, \cdots$, both bounds in (5.5) and (5.8) break down when $\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$.

To obtain an error estimate near the Stokes lines $\theta = \pm\pi$, or equivalently, $\arg z = \pm\frac{2}{3}\pi$, we use a device suggested by Boyd [6]. Rotating the path of integration in (4.23) by an angle $\eta \in \left(-\frac{\pi}{2}, 0\right)$ gives

$$R_N(\xi) = \frac{1}{2\pi}(-2\xi)^{-N}\int_0^{\infty e^{i\eta}} e^{-t}t^{N-1}\left(1+\frac{t}{2\xi}\right)^{-1} I^{(1)}\left(\frac{t}{2}\right)dt. \tag{5.10}$$

On the path of integration in (5.10), we have $\arg t = \eta$. Hence, by (5.5),

$$\left|I^{(1)}\left(\frac{t}{2}\right)\right| \le A_0\csc_1(\eta+\pi) = A_0; \tag{5.11}$$

see (5.7). The equality in (5.11) follows from the fact that $\frac{\pi}{2} < \eta+\pi < \pi$. On the other hand, we also have from (5.2)

$$\left|1+\frac{t}{2\xi}\right|^{-1} \le \csc_1(\pi+\eta-\theta). \tag{5.12}$$

A combination of (5.10), (5.11) and (5.12) yields

$$|R_N(\xi)| \le \frac{C}{(2|\xi|)^N}\csc_1(\pi+\eta-\theta)\frac{\Gamma(N)}{(\cos\eta)^N}, \tag{5.13}$$

where $C = A_0/2\pi$.

If $0 < \theta < \pi$ and θ is close to π, then for any sufficiently small $\eta \in \left(-\frac{\pi}{2}, 0\right)$ we have $-\frac{\pi}{2} < \pi+\eta-\theta < 0$. Similar, if $-\pi < \theta < 0$ and θ is close to $-\pi$, then for any sufficiently small $\eta \in \left(-\frac{\pi}{2}, 0\right)$ we have $\frac{3\pi}{2} < \pi+\eta-\theta < 2\pi$. Since the function $\csc_1(\varphi)$ defined in (5.1) and (5.3) is 2π-periodic, from (5.13) we obtain

$$|R_N(\xi)| \le \frac{C}{(2|\xi|)^N}\frac{1}{|\sin(\pm\pi+\eta-\theta)|}\frac{\Gamma(N)}{(\cos\eta)^N}, \tag{5.14}$$

where $+$ sign is taken when $\theta \nearrow \pi$ and $-$ sign is taken when $\theta \searrow -\pi$. As $\theta \to \pm\pi$, we have $\sin(\pm\pi+\eta-\theta) \sim \sin\eta$. Since η is arbitrary, we may choose it so that

$$\cos\eta \sim \left(1-\frac{1}{N}\right) \qquad \text{and} \qquad \sin\eta \sim \sqrt{\frac{2}{N}}.$$

Thus, for θ near $\pm\pi$, we can find a constant C_1 such that

$$|R_N(\xi)| \le \frac{C_1}{(2|\xi|)^N}\sqrt{\frac{N}{2}}\left(1-\frac{1}{N}\right)^{-N}\Gamma(N).$$

(An explicit value for C_1 can be given if desired.) Since $\frac{1}{x}\log(1+x) \le \frac{1}{1+x}$ for negative values of x, we have $\left(1-\frac{1}{N}\right)^{-N} \le e^{N/(N-1)}$. This together with the inequality [1]

$$\Gamma(N) < \sqrt{2\pi}N^{N-\frac{1}{2}}e^{-N+(1/12N)} \tag{5.15}$$

gives the estimate

$$|R_N(\xi)| \le C_2 e^{N(-1+\log N-\log|2\xi|)}, \tag{5.16}$$

where $C_2 = \sqrt{\pi}e^2 C_1$. The minimum value of the exponential function on the right-hand side of (5.16) is attained when

$$\frac{d}{dN}(-N+N\log N-N\log|2\xi|) = \log N - \log|2\xi| = 0.$$

Therefore, an optimal place to truncate the series in (4.12) is near

$$N = N^* := 2|\xi|.$$

With N given by this value, (5.16) yields

$$|R_N(\xi)| \le C_2 e^{-2|\xi|}, \tag{5.17}$$

as $|\xi| \to \infty$, where $\arg\xi = \theta$ is close to the Stokes lines $\theta = \pm\pi$. Applying (5.15) to the error estimate (5.8), one readily sees that inequality (5.17) holds also for ξ away from the Stokes line. Olver [9] called the expansion (4.12) with error estimate (5.17) a *uniform, exponentially improved*, asymptotic expansion in the sector $|\xi| \le \pi$. Optimatically truncated asymptotic expansions are also called super-asymptotic expansions by Berry and Howls [4]; see also [3].

6. Hyperasymptotics

Returning to (4.23), we now replace the function $I^{(1)}(t/2)$ by its asymptotic expansion (4.12). Termwise integration gives a series of integrals which can be expressed in terms of Dingle's *terminant function*

$$\int_0^\infty \frac{t^{k-1}e^{-t}}{1+t/\zeta}dt := 2\pi i(-\zeta)^k e^\zeta T_k(\zeta); \tag{6.1}$$

see Olver [9]. More precisely, we have

$$R_N(\xi) = ie^{2\xi}\sum_{r=0}^{N'-1}(-1)^r\frac{c_r}{\xi^r}T_{N-r}(2\xi) + R_{N,N'}(\xi), \tag{6.2}$$

where

$$R_{N,N'}(\xi) = \frac{1}{2\pi}(-2|\xi|)^{-N} \int_0^\infty e^{-t} t^{N-1} \left(1 + \frac{t}{2\xi}\right)^{-1} R_{N'}\left(\frac{t}{2}\right) dt. \tag{6.3}$$

The idea of re-expanding the remainder terms in optimally truncated asymptotic series was introduced by Berry and Howls [4], and they called this theory *hyperasymptotics*.

With $\arg\zeta = \phi, u = t/|\zeta|$, and $k = |\zeta| + \alpha, \alpha$ being a positive and bounded quantity, the integral in (6.1) can be written as

$$\int_0^\infty \frac{t^{k-1}e^{-t}}{1 + t/\zeta} dt = |\zeta|^k \int_0^\infty \frac{u^{\alpha-1} e^{-|\zeta|(u - \log u)}}{1 + e^{-i\phi} u} du.$$

The integrand on the right-hand side has a pole at $u = -e^{i\phi}$, which coalesces with the saddle point at $u = 1$ when $\phi = \pm\pi$. An existing theory on uniform asymptotic expansions (see [12, pp. 356-358]) can now be used to show that as $|\zeta| \to \infty$,

$$T_k(\zeta) \sim \frac{1}{2}\mathrm{erfc}(Z) - \frac{i}{\sqrt{2\pi|\zeta|}} e^{-Z^2} \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)_s g_{2s}(\phi, \alpha) \left(\frac{2}{|\zeta|}\right)^s \tag{6.4}$$

uniformly with respect to $\phi \in [-\pi + \delta, 3\pi - \delta]$, where erfc is the complementary error function, $Z := c(\phi)\sqrt{|\zeta|/2}$, and

$$\frac{1}{2}[c(\phi)]^2 := -e^{i(\phi - \pi)} + i(\phi - \pi) + 1.$$

Near $\phi = \pi$, the Taylor series of $c(\phi)$ begins

$$c(\phi) = -(\phi - \pi) - \frac{i}{6}(\phi - \pi)^2 + \frac{1}{36}(\phi - \pi)^3 + \cdots$$

The coefficients $g_{2s}(\phi, \alpha)$ can be given explicitly and the first one is

$$g_0(\phi, \alpha) = \frac{e^{i\alpha(\pi - \phi)}}{1 + e^{-i\phi}} - \frac{i}{c(\phi)}.$$

This result is taken from Olver [9]. He has shown that Z lies in the sector $-\frac{1}{4}\pi < \arg Z \le 0$ when $-\pi \le \phi \le \pi$, and in the sector $0 \le \arg(-Z) \le \frac{1}{4}\pi$ when $\pi \le \phi \le 3\pi$. As ϕ increases from values below π to values above π, Z moves from the first sector to the second sector through the origin. Since $\mathrm{erfc}(Z) = O(e^{-Z^2})$ uniformly throughout the first sector and $\mathrm{erfc}(Z) = 2 + O(e^{-Z^2})$ uniformly throughout the second sector (see [12, p.42]), if $|\zeta|$ is large and fixed and ϕ increases continuously from π^- to π^+, then (6.4) shows that $T_k(\zeta)$ changes rapidly, but smoothly, from being exponentially small to being exponentially close to one.

Coupling (4.12) and (6.2) gives

$$I^{(1)}(\xi) = \sum_{s=0}^{N-1} c_s \xi^{-s} + ie^{2\xi} \sum_{r=0}^{N'-1} (-1)^r c_r \xi^{-r} T_{N-r}(2\xi) + R_{N,N'}(\xi). \tag{6.5}$$

The remainder $R_{N,N'}(\xi)$ is given in (6.3), and can be estimated as before. Of course, it is expected to be of lower order of magnitude, and hence can be neglected. Inserting (6.4) into (6.5), we obtain from (4.5)

$$\begin{aligned} \mathrm{Ai}(z) \sim \frac{(3/2)^{1/2}}{2\pi i} z^{-1/4} \Bigg[& e^{-\xi} \sum_{s=0}^{2|\xi|-1} c_s \xi^{-s} \\ & + \frac{i}{2} \mathrm{erfc}\{c(\theta)|\xi|^{1/2}\} e^{\xi} \sum_{r=0}^{N'-1} (-1)^r c_r \xi^{-r} \Bigg], \end{aligned} \tag{6.6}$$

where $\xi = \frac{2}{3} z^{3/2}$. Note that in (6.6), we have truncated the first series at an optimal place. When θ is near π, $\mathrm{erfc}\{c(\theta)|\xi|^{1/2}\}$ will have an abrupt but smooth change. In Berry's terminology, this function is called a *Stokes multiplier*. A similar result holds for θ near $-\pi$. We shall call the abrupt but smooth change *Berry's transition*.

Exponential asymptotics of other well known entire functions can be investigated in a similar manner. However, the analysis could be considerably more complicated. For instance, in [14] and [15], Wong and Zhao have studied the Berry transition of the generalized Bessel function

$$\phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\Gamma(\rho n + \beta)},$$

where $-1 < \rho < \infty$ and β is a real or complex number. They have also discussed the exponential asymptotics of the Mittag-Leffler function [16]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \mathrm{Re}\, \alpha > 0,$$

where β may again be real or complex. These two entire functions have played an important role in the study of fractional differential equations [10]; see also [11].

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