LECTURE V

DIFFERENCE EQUATIONS AND ORTHOGONAL POLYNOMIALS

Difference equations might be a handy and practical means to compute differential equations, but they are considerably more complicated to analyze.

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1. Introduction

Orthogonal polynomials play an important role in many branches of mathematical physics; for instance, quantum mechanics, scattering theory, and statistical mechanics. A major topic in orthogonal polynomials is the study of their asymptotic behavior as the degree grows to infinity. Since the classical orthogonal polynomials (Hermite, Laguerre, and Jacobi) all satisfy a second-order linear differential equation, their asymptotic behavior can be obtained from the WKB approximation or the turning point theory (Lecture III). For discrete orthogonal polynomials (e.g., Charlier, Meixner, and Krawtchouk), one can use their generating function to obtain a Cauchy integral representation and then apply the steepest descent method or its extensions (Lecture II). However, there are orthogonal polynomials that neither satisfy any differential equation nor have integral representations. A powerful method, known as the steepest descent method for the Riemann-Hilbert problem, has recently been developed that can be applied to such polynomials. Two papers that deserve special mention are Deift & Zhou [10], and Bleher & Its [5]. In fact, there is a whole group of people now working in this area, which includes, in addition to the authors of the two above mentioned papers, Kuijlaars, Kriecherbauer, McLaughlin, Vanlessen, Venakides and Van Assche; see, e.g., [8, 9, 12, 13, 14].

However, in our view, a more desirable approach to derive asymptotic expansions for orthogonal polynomials is to develop an asymptotic theory for linear second order difference equations, in the same way as Langer, Cherry, Olver and others have done for linear second-order differential equations (Lecture III). Our view is based on the fact that any sequence of orthogonal polynomials satisfies the three-term recurrence relation

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x), \qquad n = 1, 2, \cdots, \tag{1.1}$$

where a_n, b_n and c_n are constants; see [16, p.42]. If x is a fixed number, then this recurrence relation is equivalent to the second-order linear difference equation

$$y(n+2) + n^p a(n)y(n+1) + n^q b(n)y(n) = 0, \tag{1.2}$$

where p and q are integers. When the coefficients $a(n)$ and $b(n)$ have asymptotic expansions of the form

$$a(n) = \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \qquad b(n) = \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}, \tag{1.3}$$

with $\alpha_0 \neq 0$ and $\beta_0 \neq 0$, asymptotic solutions to equation (1.2) have been given by Birkhoff [3], Birkhoff and Trjitzinsky [4]. But, their papers have been considered far too complicated and even impenetrable. A more accessible approach to these results has been given in more recent years by Wong and Li [23, 24]. However, the results in all these papers can be applied to (1.1) only when x is a fixed number. When x is a parameter and allowed to vary, not much work has been done in this area until just recently. In a series of papers [19, 20, 21], Wang and Wong have derived asymptotic expansions for the solutions to (1.1), which hold uniformly for x in infinite intervals. They first define a sequence $\{K_n\}$ recursively by $K_{n+1}/K_{n-1} = c_n$, with K_0 and K_1 depending on the particular sequence of polynomials. Then they put $A_n \equiv a_n K_n/K_{n+1}, B_n \equiv b_n K_n/K_{n+1}$ and $P_n(x) \equiv p_n(x)/K_n$, so that (1.1) becomes

$$P_{n+1}(x) - (A_n x + B_n)P_n(x) + P_{n-1}(x) = 0. \tag{1.4}$$

The coefficients A_n and B_n are assumed to have asymptotic expansions of the form

$$A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s} \qquad \text{and} \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}, \tag{1.5}$$

where θ is a real number and $\alpha_0 \neq 0$.

In this lecture, we summarize the results in [23, 24] and [20, 21]. More precisely, in Sec. 2 we present the results for equation (1.2) when $p = q = 0$, and discuss the general case in Sec. 3. Equation (1.4) is studied in Sec. 4 when $\theta \neq 0$, and in Sec. 5 when $\theta = 0$. Interested readers are referred to the original papers for proofs of the results.

2. Normal and subnormal series

When $p = q = 0$, equation (1.2) becomes

$$y(n+2) + a(n)y(n+1) + b(n)y(n) = 0. \tag{2.1}$$

Asymptotic solutions to this equation are classified by the roots of the *characteristic equation*

$$\rho^2 + a_0\rho + b_0 = 0. \tag{2.2}$$

Two possible values of ρ are

$$\rho_1, \rho_2 = -\frac{1}{2}a_0 \pm \left(\frac{1}{4}a_0^2 - b_0\right)^{1/2}. \tag{2.3}$$

If $\rho_1 \neq \rho_2$, i.e., $a_0^2 \neq 4b_0$, then Birkhoff [2] showed that (2.1) has two linearly independent solutions $y_j(n), j = 1, 2$, such that

$$y_j(n) \sim \rho_j^n n^{\alpha_j} \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^s}, \qquad n \to \infty, \tag{2.4}$$

where

$$\alpha_j = -\frac{a_1\rho_j + b_1}{2\rho_j^2 + \rho_j a_0} = \frac{a_1\rho_j + b_1}{a_0\rho_j + 2b_0}, \tag{2.5}$$

$c_{0,j} = 1$ and

$$\begin{aligned} \rho_j(a_0 + 2\rho_j)sc_{s,j} = \sum_{r=1}^{s} \Bigg[& \rho_j^2 2^{r+1} \binom{\alpha_j + r - s}{r+1} \\ & + \rho_j \sum_{q=0}^{r+1} \binom{\alpha_j + r - s}{r+1-q} a_q + b_{r+1} \Bigg] c_{s-r,j}, \end{aligned} \tag{2.6}$$

$s = 1, 2, \cdots$. This construction fails when and only when $\rho_1 = \rho_2$, i.e., when $a_0^2 = 4b_0$. The series in (2.4) are known as *normal series* or *normal solutions*.

If $\rho_1 = \rho_2$ but their common value $\rho = -\frac{1}{2}a_0$ is not a root of the auxiliary equation

$$a_1\rho + b_1 = 0, \tag{2.7}$$

i.e., $2b_1 \neq a_0a_1$, then Adams [1] showed that (2.1) has two linearly independent solutions $y_j(n), j = 1, 2$, such that

$$y_j(n) \sim \rho^n \exp((-1)^j \gamma\sqrt{n}) n^{\alpha} \sum_{s=0}^{\infty} (-1)^{js} \frac{c_s}{n^{s/2}}, \tag{2.8}$$

where

$$\gamma = 2\sqrt{\frac{a_0a_1 - 2b_1}{2b_0}}, \tag{2.9}$$

$$\alpha = \frac{1}{4} + \frac{b_1}{2b_0}, \tag{2.10}$$

and $c_0 = 1$. Series of the form (2.8) are called *subnormal series* or *subnormal solutions*. Higher coefficients can be determined by formal substitution.

When the double root of the characteristic equation (2.2) satisfies the auxiliary equation (2.7), i.e., when $2b_1 = a_0a_1$, we have three (exceptional) cases to consider, depending on the values of the zeros $\alpha_1, \alpha_2 (\mathrm{Re}\ \alpha_2 \geq \mathrm{Re}\ \alpha_1)$ of the *indicial polynomial*

$$q(\alpha) = \alpha(\alpha - 1)\rho^2 + (a_1\alpha + \alpha_2)\rho + b_2. \tag{2.11}$$

Case (i) : $\alpha_2-\alpha_1 \neq 0, 1, 2, \cdots$. In this case, (2.1) has independent solutions $y_j(n), j = 1, 2$, of the form

$$y_j(n) \sim \rho^n n^{\alpha_j} \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^s}, \qquad n \to \infty, \tag{2.12}$$

with $c_{0,j} = 1$.

Case (ii) : $\alpha_2 - \alpha_1 = 1, 2, \cdots$. Here, (2.12) applies only in the case of $j = 1$. A second independent solution is given by

$$y_2(n) \sim \rho^n n^{\alpha_2} \sum_{s=0}^{\infty}{}' \frac{d_s}{n^s} + c(\log n) y_1(n), \tag{2.13}$$

where the prime on $\sum$ denotes that the term for $s = \alpha_2 - \alpha_1$ is absent. The coefficients c and d_s can be determined by formal substitution, beginning with $d_0 = 1$.

Case (iii) : $\alpha_2 = \alpha_1$. As in case (ii), (2.12) again gives only one solution $y_1(n)$. The second solution is given by

$$y_2(n) \sim \rho^n n^{\alpha_2} \sum_{s=1}^{\infty} \frac{d_s}{n^s} + (\log n) y_1(n). \tag{2.14}$$

3. Equation (1.2) with p and $q \neq 0$.

Many orthogonal polynomials satisfy difference equations of the form (1.2), but not of the form (2.1). For example, the recurrence relation for the Charlier polynomials is

$$C_{n+1}^{(a)}(x) + (n + a - x) C_n^{(a)}(x) + anC_{n-1}^{(a)}(x) = 0, \qquad a \neq 0, \tag{3.1}$$

and the recurrence relation for the Bessel polynomials $y_n(x)$ is

$$y_{n+1}(x) = (2n + 1)xy_n(x) + y_{n-1}(x); \tag{3.2}$$

see [6, Chap.VI].

If the exponents p and q in (1.2) are related in the manner $q = 2p$, then (1.2) can be reduced to (2.1) by using the transformation

$$x(n) = [(n - 2)!]^{\mu} y(n). \tag{3.3}$$

Indeed, substitution of (3.3) in (1.2) gives the equivalent equation

$$x(n + 2) + n^{p+\mu} a(n) x(n + 1) + n^{q+2\mu} b^*(n) x(n) = 0, \tag{3.4}$$

where

$$b^*(n) = b(n) \left(1 - \frac{1}{n}\right)^{\mu} = \sum_{s=0}^{\infty} \frac{b_s^*}{n^s}. \tag{3.5}$$

(Note that the constant term in the expansion (3.5) is again not zero.) If $q = 2p$ then, by taking $\mu = -p$, (3.4) becomes an equation of the same form as (2.1). Thus, our discussion of equation (1.2) consists of only two cases, namely, (i) $q < 2p$ and (ii) $q > 2p$. We set $k \equiv 2p - q$. Each of these two cases has two or three subcases, depending on the values of k.

In case (i), i.e., when $k > 0$, it has been shown in [23] that equation (1.2) has an asymptotic solution of the form

$$y_1(n) \sim [(n-2)!]^p \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^s}, \tag{3.6}$$

where $\rho = -a_0, \alpha = a_1/a_0$ if $k > 1$, and

$$\alpha = \frac{1}{a_0}\left(a_1 - \frac{b_0}{a_0}\right) \qquad \text{if} \quad k = 1. \tag{3.7}$$

A second independent solution is given by

$$y_2(n) = [(n-2)!]^{q-p} \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{d_s}{n^s}, \tag{3.8}$$

where $\rho = -b_0/a_0$,

$$\alpha = \frac{b_0}{a_0^2} - \frac{a_1}{a_0} + \frac{b_1}{b_0} - p + q \qquad \text{if} \quad k = 1, \tag{3.9}$$

and

$$\alpha = \frac{b_1}{b_0} - \frac{a_1}{a_0} - p + q \qquad \text{if} \quad k > 1. \tag{3.10}$$

Recursive formulas can be obtained for the coefficients c_s and d_s by formal substitution.

In case (ii), i.e., when $k < 0$, we have three subcases to consider depending on whether $k = -1$, or $k \leq -3$ and is odd, or $k \leq -2$ and is even. If $k = -1$, then two asymptotic solutions are of the form

$$y_j(n) \sim [(n-2)!]^{q/2} \rho_j^n \exp(\gamma_j \sqrt{n}) n^\alpha \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^{s/2}}, \tag{3.11}$$

$j = 1, 2$, where $\rho_j^2 = -b_0, \gamma_j = -a_0/\rho_j$ and

$$\alpha = \frac{b_1}{2b_0} + \frac{q}{4}. \tag{3.12}$$

If $k \leq -3$ and k is odd, then the exponential factor in (3.11) is absent and equation (3.11) becomes

$$y_j(n) \sim [(n-2)!]^{q/2} \rho_j^n n^\alpha \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^{s/2}}, \tag{3.13}$$

where ρ and α are as given in (3.11) and (3.12). If $k \leq -2$ and k is even, then coefficients of odd powers of $n^{-\frac{1}{2}}$ all vanish and (3.11) simplifies to

$$y_j(n) \sim [(n-2)!]^{q/2} \rho_j^n n^{\alpha_j} \sum_{s=0}^{\infty} \frac{c_{s,j}}{n^s} \tag{3.14}$$

with $\rho_j^2 = -b_0 (j = 1, 2), \alpha_j = \alpha$ given in (3.12) if $k = -4, -6, \cdots$, and

$$\alpha_j = -\frac{1}{2\rho_j^2}\left(b_1 + \frac{1}{2} q b_0 + \rho_j a_0\right) \qquad \text{if} \quad k = -2. \tag{3.15}$$

4. Airy-type Expansion

Let τ_0 be a constant, and put $\nu := n + \tau_0$. Clearly, the expansions in (1.5) can be recast in the form

$$A_n \sim \nu^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha'_s}{\nu^s} \qquad \text{and} \qquad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{\nu^s}. \tag{4.1}$$

In (1.4), we now let $x = \nu^\theta t$ and $P_n = \lambda^n$. Substituting (4.1) into (1.4) and letting $n \to \infty$ (and hence $\nu \to \infty$), we obtain the characteristic equation

$$\lambda^2 - (\alpha'_0 t + \beta'_0)\lambda + 1 = 0. \tag{4.2}$$

The roots of this equation are given by

$$\lambda = \frac{1}{2}\left[(\alpha'_0 t + \beta'_0) \pm \sqrt{(\alpha'_0 t + \beta'_0)^2 - 4}\right], \tag{4.3}$$

and they coincide when $t = t_\pm$, where

$$\alpha'_0 t_\pm + \beta'_0 = \pm 2. \tag{4.4}$$

The values $t_\pm$ play an important role in the asymptotic theory of the three-term recurrence relation (1.4), and they correspond to the transition points (i.e., turning points and poles) occurring in differential equations; cf. [15, p.362]. For this reason, we shall also call them *transition points*. Since t_+ and t_- have different values, we may restrict ourselves to just the case $t = t_+$. For t near t_+, we try a formal series solution to (1.4) in the form

$$P_n(x) = \sum_{s=0}^{\infty} \chi_s(\xi) \delta^s, \tag{4.5}$$

where δ is a small quantity depending on ν (e.g., a power of ν^{-1}) and ξ depends on x and ν. This particular form of expansion was suggested by Costin and Costin [7]. In terms of the exponent θ in (4.1) and the transition point t_+,

we have three cases to consider; namely, (i) $\theta \neq 0$ and $t_+ \neq 0$; (ii) $\theta \neq 0$ and $t_+ = 0$; and (iii) $\theta = 0$.

In this section, we shall consider only the first case, namely, case (i). For simplicity, we assume $\theta > 0$. The analysis for the case $\theta < 0$ is very similar; for an important example with $\theta = -1$, the interested reader is referred to [19]. In case (i), we choose

$$\tau_0 = -\frac{(\alpha_1 t_+ + \beta_1)}{(2-\beta_0)\theta} \tag{4.6}$$

so that

$$\alpha_1' t_+ + \beta_1' = 0. \tag{4.7}$$

Also, in (4.5), we choose $\delta = \nu^{-\frac{1}{3}}$ and $\xi = \delta^{-2}\zeta(t)$, where $\zeta(t)$ is an increasing function with $\zeta(t_+) = 0$. Substituting (4.5) into (1.4), we find that χ_0 satisfies the Airy equation

$$\chi''(\xi) = \Theta^3 \xi \chi(\xi), \tag{4.8}$$

where $\Theta^3 = \alpha_0'/\theta^2 t_+^2 \zeta'^3(t_+)$, and that each $\chi_s, s = 1, 2, \cdots$, is a solution of an inhomogeneous Airy equation. (For details, see [18, Chap.3].) This suggests that instead of (4.5), we might as well try the more accurate formal series solution

$$\begin{aligned} P_n(\nu^\theta t) = \nu^{\frac{1}{6}} \left(\frac{4\zeta}{(\alpha_0' t + \beta_0')^2 - 4} \right)^{\frac{1}{4}} \Bigg[& \mathrm{Ai}(\nu^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\nu^s} \\ & + \frac{\mathrm{Ai}'(\nu^{\frac{2}{3}}\zeta)}{\nu^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\nu^s} \Bigg], \end{aligned} \tag{4.9}$$

which we have encountered in the differential equation theory [15, p.409] and the integral approach [22, p.370]. It turns out that this form of solution is not sufficiently general, unless

$$\alpha_1' = \beta_1' = 0. \tag{4.10}$$

It is interesting to note that this condition holds in most of the classical cases. In fact, in [11] Dingle and Morgan have assumed that $\alpha_{2s+1} = \beta_{2s+1} = 0$ for $s = 0, 1, 2, \cdots$. For a more general form of the solution, we refer the reader to [20]. To show that (4.9) is indeed an asymptotic expansion, we need first to determine the function $\zeta(t)$ in the argument of the Airy function. To do this, we first substitute (4.9) in (1.4), then match the coefficients of Ai and Ai$'$, and finally let $\nu \to \infty$. This leads us to

$$\begin{aligned} \frac{2}{3}[\zeta(t)]^{\frac{3}{2}} = \alpha_0' t^{1/\theta} \int_{t_+}^{t} & \frac{s^{-1/\theta}}{\sqrt{(\alpha_0' s + \beta_0')^2 - 4}}\, ds \\ & - \log \frac{\alpha_0' t + \beta_0' + \sqrt{(\alpha_0' t + \beta_0')^2 - 4}}{2} \end{aligned} \tag{4.11}$$

if $t \geq t_+$, and

$$\begin{aligned}\frac{2}{3}[-\zeta(t)]^{\frac{3}{2}} =& \cos^{-1}\frac{\alpha_0' t + \beta_0'}{2} \\ &- \alpha_0' \, t^{1/\theta} \int_t^{t_+} \frac{s^{-1/\theta}}{\sqrt{4-(\alpha_0' s + \beta_0')^2}}\, ds\end{aligned} \tag{4.12}$$

if $t < t_+$. To make the presentation simpler, we have assumed in [20] that $t_- < 0 < t_+$. This assumption is equivalent to the condition $|\beta_0| < 2$.

The second solution, independent of (4.9), is given by

$$\begin{aligned}Q_n(\nu^\theta t) \sim \nu^{\frac{1}{6}} \left(\frac{4\zeta}{(\alpha_0' t + \beta_0')^2 - 4}\right)^{\frac{1}{4}} \Bigg[&\mathrm{Bi}(\nu^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\nu^s} \\ &+ \frac{\mathrm{Bi}'(\nu^{\frac{2}{3}}\zeta)}{\nu^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\nu^s} \Bigg].\end{aligned} \tag{4.13}$$

The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ are determined successively from some recursive formulas, beginning with $A_0(\zeta) = 1$ and $\zeta^{\frac{1}{2}} B_0(\zeta) = 0$.

5. Bessel-type Expansion

We now consider case (iii), i.e., $\theta = 0$ in (4.1). As in Sec. 4, we let τ_0 be a constant and define $N := n + \tau_0$. The characteristic equation (4.2) is obtained in the same manner, except that t is replaced by x. The characteristic roots again coincide when $x = x_\pm$, where $\alpha_0 x_\pm + \beta_0 = \pm 2$. For x near x_+, we try a formal series solution of the form

$$P_n(x) = \sum_{s=0}^{\infty} \chi_s(\xi) N^{-s}, \tag{5.1}$$

where ξ depends on x and N. In the present case, we choose $\xi = N\zeta^{1/2}(x)$, where $\zeta(x)$ is an increasing function with $\zeta(x_+) = 0$. Substituting (5.1) into (1.4), one finds that $\chi_0(\xi)$ satisfies the Bessel equation

$$\frac{d^2\chi_0}{d\xi^2} = \left(\frac{\alpha_0'}{\zeta'(0)} + \frac{\alpha_2' x_+ + \beta_2'}{\xi^2}\right)\chi_0.$$

Thus, it follows that $\chi_0(\xi)$ can be expressed in terms of either the Bessel functions $J_\nu(\xi)$ and $Y_\nu(\xi)$, or the modified Bessel functions $I_\nu(\xi)$ and $K_\nu(\xi)$. That is, there are constants C_1 and C_2 such that

$$\chi_0(\xi) = C_1 \xi^{1/2} J_\nu(\xi) + C_2\, \xi^{1/2} Y_\nu(\xi) \qquad \text{if} \quad \alpha_0' < 0$$

and

$$\chi_0(\xi) = C_1 \xi^{1/2} I_\nu(\xi) + C_2\, \xi^{1/2} K_\nu(\xi) \qquad \text{if} \quad \alpha_0' > 0,$$

where

$$\nu = \left(\alpha_2' x_+ + \beta_2' + \frac{1}{4}\right)^{1/2} \tag{5.2}$$

and we take the square root with nonnegative real part. Moreover, each of the subsequent coefficient functions $\chi_s(\xi), s = 1, 2, \cdots$, in (5.1) satisfies an inhomogeneous Bessel equation. This suggests that instead of (5.1), we might as well try the formal series solution of the form

$$P_n(x) = N^{\frac{1}{2}} Z_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{\overline{A}_s(\zeta)}{N^s} + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} Z_{\nu+1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{\overline{B}_s(\zeta)}{N^s} \tag{5.3}$$

motivated from the difference equation theory [15, p.441]. In (5.3), $Z_\nu(\xi)$ can be any solution of the modified Bessel equation

$$y'' + \frac{1}{x} y' - \left(1 + \frac{\nu^2}{x^2}\right) y = 0. \tag{5.4}$$

The function $\zeta(x)$ in (5.3) can be determined by subtituting (5.3) in (1.4), matching the coefficients of Z_ν and $Z_{\nu+1}$, and letting $n \to \infty$. The result is

$$\zeta^{\frac{1}{2}}(x) = \cosh^{-1}\left(\frac{\alpha_0' x + \beta_0'}{2}\right). \tag{5.5}$$

In [21], it was shown that when $\theta = 0$ in (4.1), equation (1.4) has a pair of linearly independent solutions

$$\begin{aligned} P_n(x) \sim \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} \Bigg[& N^{\frac{1}{2}} I_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \\ & + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} I_{\nu-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \Bigg] \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} Q_n(x) \sim \left(\frac{4\zeta}{(\alpha_0' x + \beta_0')^2 - 4}\right)^{\frac{1}{4}} \Bigg[& N^{\frac{1}{2}} K_\nu(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{N^s} \\ & + N^{\frac{1}{2}} \zeta^{\frac{1}{2}} K_{\nu-1}(N\zeta^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{N^s} \Bigg], \end{aligned} \tag{5.7}$$

where ν is given in (5.2) and

$$N = n + \tau_0 = n - \frac{\alpha_3 x_+ + \beta_3}{2\left(\nu^2 - \frac{1}{4}\right)}. \tag{5.8}$$

References

1. C. R. Adams, *On the irregular cases of linear ordinary difference equations*, Trans. Amer. Math. Soc. **30** (1928), 507-541.
2. G. D. Birkhoff, *General theory of linear difference equations*, Trans. Amer. Math. Soc. **12** (1911), 243-284.
3. G. D. Birkhoff, *Formal theory of irregular linear difference equations*, Acta Math. **54** (1930), 205-246.
4. G. D. Birkhoff and W. J. Trjitzinsky, *Analytic theory of singular difference equations*, Acta Math. **60** (1932), 1-89.
5. P. Bleher and A. Its, *Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model*, Ann. Math. **150** (1999), 185-266.
6. T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
7. O. Costin and R. Costin, *Rigorous WKB for finite-order linear recurrence relations with smooth coefficients*, SIAM J. Math. Anal. (1996), 110-134.
8. P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Comm. Pure Appl. Math. **52** (1999), 1335-1425.
9. P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Comm. Pure Appl. Math. **52** (1999), 1491-1552.
10. P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems, Applications for the MKdV equation*, Ann. Math. **137** (1993), 295-368.
11. R. B. Dingle and G. J. Morgan, *WKB methods for difference equtions* I, Appl. Sci. Res. **18** (1967), 221-237.
12. T. Kriecherbauer and K. T-R McLaughlin, *Strong asymptotics of polynomials orthogonal with respect to Freud weights*, Internat. Math. Res. Notes (1999), 299-333.
13. A. B. J. Kuijlaars and K. T-R McLaughlin, *Riemann-Hilbert analysis for Laguerre polynomials with large negative parameter*, Comput. Math. Funct. Theory **1** (2001), 205-233.
14. A. B. J. Kuijlaars and M. Vanlessen, *Universality for eigenvalue correlations from the modified Jacobi unitary ensemble*, Internat. Math. Res. Notices (2002), 1575-1600.
15. F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974. (Reprinted by A. K. Peters Ltd., Wellesley, 1997).
16. G. Szegö, *Orthogonal Polynomials*, Fourth edition, Colloquium Publications, Vol 23, Amer. Math. Soc., Providence R. I., 1975.
17. W. Van Assche, J. S. Geromino, and A. B. J. Kuijlaars, *Riemann-Hilbert problems for multiple orthogonal polynomials*, pp.23-50 in "NATO ASI Special Functions 2000" (J. Bustoz et. al., eds.), Kluwer Academic Publishers, Dordrecht 2001.
18. Z. Wang, *Asymptotic expansions for second order linear difference equations with a turning point*, Ph.D. Thesis, City University of Hong Kong, 2001.
19. Z. Wang and R. Wong, *Uniform asymptotic expansion of $J_\nu(\nu a)$ via a difference equation*, Numer. Math. **91** (2002), 147-193.
20. Z. Wang and R. Wong, *Asymptotic expansions for second-order linear difference equations with a turning point*, Numer. Math. **94** (2003), 147-194.
21. Z. Wang and R. Wong, *Linear difference equations with transition points*, Math. Comp., to appear.
22. R. Wong, *Asymptotic Approximations of Integrals*, Academic Press, Boston, 1989. (Reprinted by SIAM, Philadelphia, PA, 2001.)
23. R. Wong and H. Li, *Asymptotic expansions for second-order linear difference equations*, J. Comput. Appl. Math. **41** (1992), 65-94.
24. R. Wong and H. Li, *Asymptotic expansions for second-order linear difference equations* II, Stud. Appl. Math. **87** (1992), 289-324.