LECTURE I

STATIONARY PHASE APPROXIMATION

> Kelvin invented his method of stationary phase as a tool for approximating the solution of this particular problem, and it is indeed a beautiful and strikingly successful example of its usefulness.
>
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1. Introduction

According to Watson [13, p.229-230], the principle of stationary phase was first introduced explicitly by Lord Kelvin in 1887, although the essence of this principle can be found in the earlier works of Cauchy, Stokes and Riemann. Kelvin's original problem was to find the asymptotic behavior of the integral

$$u(x) = \frac{1}{2\pi}\int_0^\infty \cos\{m[x - tf(m)]\}dm, \qquad \text{as } x \to \infty,$$

where all variables are real. However, it is now more common to state the principle in terms of the integral

$$I(\lambda) = \int_a^b g(t)e^{i\lambda f(t)}dt, \tag{1.1}$$

where a, b and the function $f(t)$ are real, and λ is a large real parameter.

The underlying principle of stationary phase is the assertion that the major asymptotic contribution to the integral (1.1) comes from points where the *phase function* $f(t)$ is *stationary*, i.e., where $f'(t)$ vanishes. The function $g(t)$ is known as the *amplitude function* in wave propagation theory. If $f(t)$ has only one stationary point, say c, in (a, b), then the principle states that asymptotically we have

$$I(\lambda) \sim g(c)\sqrt{\frac{2\pi}{\lambda|f''(c)|}}e^{i[\lambda f(c)+\sigma\pi/4]}, \tag{1.2}$$

as $\lambda \to +\infty$, where $\sigma = \operatorname{sgn} f''(c)$. A heuristic argument for (1.2) is given in Sec. 2, followed by a rigorous proof based on the Abel summability method introduced by Olver [8] in 1974.

In Sec. 3, we consider a case in which the phase function $f(x)$ involves an auxiliary parameter α, and present an asymptotic formula which holds uniformly when two stationary points of $f(x)$ coalesce as α tends to a critical value, say $\alpha = \alpha_0$.

The two-dimensional generalization of (1.1) is the double integral

$$I(\lambda) = \iint_D g(x,y)e^{i\lambda f(x,y)}dxdy, \tag{1.3}$$

where D is a bounded domain, $f(x,y)$ is a real-valued function, and λ is a large positive parameter. If $f(x,y)$ has only one stationary point, say (x_0,y_0), then the analogue of (1.2) is

$$\begin{aligned} I(\lambda) \sim & \frac{2\pi}{\lambda}|\det f''(x_0,y_0)|^{-\frac{1}{2}}g(x_0,y_0) \\ & \times \exp\left\{i\lambda f(x_0,y_0) + i\frac{\pi}{4}\sigma_f(x_0,y_0)\right\}, \end{aligned} \tag{1.4}$$

where

$$\det f''(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0) \tag{1.5}$$

and

$$\sigma_f(x_0,y_0) = \begin{cases} 2 & \text{if } \det f''(x_0,y_0) > 0 \text{ and } f_{xx}(x_0,y_0) > 0, \\ -2 & \text{if } \det f''(x_0,y_0) > 0 \text{ and } f_{xx}(x_0,y_0) < 0, \\ 0 & \text{if } \det f''(x_0,y_0) < 0. \end{cases} \tag{1.6}$$

We will again first give a heuristic derivation of (1.4) in Sec. 4, and then a rigorous demonstration based on the method of resolution of multiple integrals used by Jones and Kline [6].

The final section (Sec. 5) contains a two-dimensional generalization of the problem discussed in Sec. 3; that is, a problem in which the phase function $f(x,y)$ in the double integral (1.3) involves a parameter α, and has two simple stationary points in the integration domain D, which coalesce as α approaches a critical value α_0. An asymptotic formula is given, which holds uniformly in a neighborhood of α_0.

2. Stationary-Phase Approximation

Returning to the integral (1.1), we suppose that $f(c)$ is a simple minimum in the interval (a,b), i.e., $f'(c) = 0$ and $f''(c) > 0$. Then, according to Kelvin's principle, we have heuristically

$$\begin{aligned} I(\lambda) \sim & \int_{c-\varepsilon}^{c+\varepsilon} g(t)e^{i\lambda f(t)}dt \qquad (\varepsilon > 0) \\ \sim & g(c)\int_{c-\varepsilon}^{c+\varepsilon} e^{i\lambda[f(c)+f''(c)(t-c)^2/2]}dt \\ \sim & g(c)e^{i\lambda f(c)}\int_{-\infty}^{\infty} e^{i\lambda f''(c)(t-c)^2/2}dt \\ \sim & g(c)\sqrt{\frac{2\pi}{\lambda f''(c)}}e^{i[\lambda f(c)+\pi/4]}. \end{aligned} \tag{2.1}$$

Similarly, if $f(c)$ is a maximum, i.e., $f'(c) = 0$ and $f''(c) < 0$, then

$$I(\lambda) \sim g(c)\sqrt{\frac{2\pi}{-\lambda f''(c)}}e^{i[\lambda f(c)-\pi/4]}. \tag{2.2}$$

By coupling (2.1) and (2.2), one obtains the stationary-phase approximation (1.2)

To establish (2.1) rigorously, we subdivide the integration interval (a, b) at the stationary point $t = c$, so that

$$I(\lambda) = \left(\int_a^c + \int_c^b\right) g(t)e^{i\lambda f(t)}dt := I_1(\lambda) + I_2(\lambda). \tag{2.3}$$

For the moment, we consider only the second integral in which we introduce the variable u defined by

$$u := f(t) - f(c) = \sum_{s=0}^{\infty} f_s(t-c)^{s+2}, \tag{2.4}$$

and put

$$\rho = f(b) - f(c). \tag{2.5}$$

For convenience, we shall assume that ρ is finite. The treatment of the case in which ρ is infinite is actually simpler. In terms of the new variable, the integral $I_2(\lambda)$ becomes

$$I_2(\lambda) = e^{i\lambda f(c)} \int_0^{\rho} \varphi(u)e^{i\lambda u}du, \tag{2.6}$$

where

$$\varphi(u) = g(t)/f'(t). \tag{2.7}$$

For small $u, t - c$ and $\varphi(u)$ can be expanded in series of the form

$$t - c = \sum_{s=1}^{\infty} c_s u^{s/2}, \qquad \varphi(u) = \sum_{s=0}^{\infty} a_s u^{(s-1)/2}. \tag{2.8}$$

Since $f_0 = f''(c)/2$, it is readily verified that

$$a_0 = \frac{g(c)}{[2f''(c)]^{1/2}}. \tag{2.9}$$

For each $n \geq 0$, define $\varphi_n(u)$ by

$$\varphi(u) = \sum_{s=0}^{n-1} a_s u^{(s-1)/2} + \varphi_n(u). \tag{2.10}$$

Clearly,

$$\varphi_n(u) \sim a_n u^{(n-1)/2} + a_{n+1} u^{n/2} + \cdots, \qquad u \to 0^+, \tag{2.11}$$

and

$$\varphi_n^{(j)}(u) = \varphi^{(j)}(u) - \sum_{s=0}^{n-1} \frac{\Gamma[(s+1)/2]}{\Gamma[(s+1-2j)/2]} a_s u^{(s-1-2j)/2}, \tag{2.12}$$

$j = 0, 1, 2, \cdots$.

Now we quote a few results from the Abel summability theory. First, for any piecewise continuous function ψ in $(0, \infty)$, we have

$$\lim_{\varepsilon \to 0^+} \int_0^\infty e^{-\varepsilon u} \psi(u) du = \int_0^\infty \psi(u) du, \tag{2.13}$$

if the integral on the right exists as an improper Riemann integral; see [12, p.26]. Second, for $\lambda > 0$ and Re $\alpha > 0$,

$$\lim_{\varepsilon \to 0^+} \int_0^\infty e^{-\varepsilon u} e^{i\lambda u} u^{\alpha-1} du = \frac{e^{\alpha \pi i/2} \Gamma(\alpha)}{\lambda^\alpha}. \tag{2.14}$$

Since the integral on the left-hand side of (2.14) does not exist when we set $\varepsilon = 0$, this result does not follow from (2.13). However, it can be proved directly, by expressing the integral in terms of the gamma function and then using the continuity of this function. Also, when $\lambda > 0$ and $\rho > 0$,

$$\lim_{\varepsilon \to 0^+} \int_\rho^\infty e^{-\varepsilon u} e^{i\lambda u} u^{\alpha-1} du = \frac{e^{\alpha \pi i/2}}{\lambda^\alpha} \Gamma(\alpha, -i\lambda\rho), \tag{2.15}$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function.

Substituting (2.10) into (2.6) and integrating term by term in the sense of Abel summability (2.13), we obtain from (2.13) and (2.14)

$$\int_0^\rho \varphi(u) e^{i\lambda u} du = \sum_{s=0}^{n-1} a_s \Gamma\left(\frac{s+1}{2}\right) \frac{e^{i(s+1)\pi/4}}{\lambda^{(s+1)/2}} + E_n(\lambda) - F_n(\lambda), \tag{2.16}$$

where

$$E_n(\lambda) = \lim_{\varepsilon \to 0^+} \int_0^\rho e^{-(\varepsilon - i\lambda)u} \varphi_n(u) du \tag{2.17}$$

and

$$F_n(\lambda) = \sum_{s=0}^{n-1} a_s \lim_{\varepsilon \to 0} \int_\rho^\infty e^{-(\varepsilon - i\lambda)u} u^{(s-1)/2)} du. \tag{2.18}$$

By repeated integration by parts, we have

$$\begin{aligned} E_n(\lambda) = \left(\frac{i}{\lambda}\right)^m \varphi_n^{(m-1)}(0) - e^{i\lambda\rho} \sum_{j=0}^{m-1} \varphi_n^{(j)}(\rho) \left(\frac{i}{\lambda}\right)^{j+1} \\ + \left(\frac{i}{\lambda}\right)^m \int_0^\rho e^{i\lambda u} \varphi_n^{(m)}(u) du, \end{aligned} \tag{2.19}$$

where $m = [(n+1)/2]$; i.e., $m = n/2$ if n is even, and $m = (n+1)/2$ if n is odd. Note that $\varphi_n^{(m-1)}(0) = 0$, when n is even.

To deal with $F_n(\lambda)$, we use (2.15) and obtain

$$F_n(\lambda) = \sum_{s=0}^{n-1} a_s \frac{e^{i(s+1)\pi/4}}{\lambda^{(s+1)/2}} \Gamma\left(\frac{s+1}{2}, -i\lambda\rho\right). \tag{2.20}$$

From the well known integral representation

$$\Gamma(\alpha, z) = \int_z^\infty e^{-t} t^{\alpha-1} dt, \tag{2.21}$$

we have by repeated integration by parts

$$\Gamma(\alpha, z) = e^{-z} z^{\alpha-1} \sum_{j=0}^{m-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha-j)} \frac{1}{z^j} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} \Gamma(\alpha-m, z). \tag{2.22}$$

Hence

$$\begin{aligned} F_n(\lambda) = e^{i\lambda\rho} \sum_{s=0}^{n-1} \sum_{j=0}^{m-1} a_s \frac{\Gamma[(s+1)/2]}{\Gamma[(s+1-2j)/2]} \left(\frac{i}{\lambda}\right)^{j+1} \rho^{(s-1-2j)/2} \\ + \sum_{s=0}^{n-1} a_s \frac{e^{i(s+1)\pi/4}}{\lambda^{(s+1)/2}} \frac{\Gamma[(s+1)/2]}{\Gamma[(s+1-2m)/2]} \\ \times \Gamma\left(\frac{s+1-2m}{2}, -i\lambda\rho\right). \end{aligned} \tag{2.23}$$

Subtracting (2.23) from (2.19), and making use of (2.12), we arrive at

$$\begin{aligned} E_n(\lambda) - F_n(\lambda) = -e^{i\lambda\rho} \sum_{j=0}^{m-1} \varphi^{(j)}(\rho) \left(\frac{i}{\lambda}\right)^{j+1} + \left(\frac{i}{\lambda}\right)^m \varphi_n^{(m-1)}(0) \\ + \{\delta_n(\lambda) - \varepsilon_n(\lambda)\}, \end{aligned} \tag{2.24}$$

where

$$\delta_n(\lambda) = \left(\frac{i}{\lambda}\right)^m \int_0^\rho e^{i\lambda u} \varphi_n^{(m)}(u) du \tag{2.25}$$

and

$$\begin{aligned} \varepsilon_n(\lambda) = \sum_{s=0}^{n-1} e^{i(s+1)\pi/4} \frac{\Gamma[(s+1)/2]}{\Gamma[(s+1-2m)/2]} \\ \times \Gamma\left(\frac{s+1-2m}{2}, -i\lambda\rho\right) \frac{a_s}{\lambda^{(s+1)/2}}. \end{aligned} \tag{2.26}$$

Recall that $\varphi_n^{(m-1)}(0) = 0$ when n is even. When n is odd, $m = (n+1)/2$ and we have $\varphi_n^{(m-1)}(0) = (m-1)!\, a_n$. Thus, by coupling (2.16) with (2.24), and with $m = [(n+1)/2]$, we can write

$$\begin{aligned}\int_0^\rho \varphi(u)e^{i\lambda u}du = &\sum_{s=0}^{n-\nu} a_s \Gamma\left(\frac{s+1}{2}\right)\frac{e^{i(s+1)\pi/4}}{\lambda^{(s+1)/2}} \\ &- e^{-i\lambda\rho}\sum_{j=0}^{m-1}\varphi^{(j)}(\rho)\left(\frac{i}{\lambda}\right)^{j+1} + \{\delta_n(\lambda) - \varepsilon_n(\lambda)\},\end{aligned} \tag{2.27}$$

where $\nu = 0$ when n is odd (i.e., $2m = n+1$), and $\nu = 1$ when n is even (i.e., $2m = n$).

To estimate the error term in (2.27), we first have from (2.25)

$$|\delta_n(\lambda)| \le \frac{1}{\lambda^m}\int_0^\rho |\varphi_n^{(m)}(u)|du. \tag{2.28}$$

By deforming the integration path in (2.21) until it lies on the imaginary axis, it can be shown upon a partial integration that

$$|\Gamma(\alpha, \pm iy)| \le 2y^{\alpha-1}, \qquad \alpha < 1, \qquad y > 0. \tag{2.29}$$

Hence we also have

$$|\varepsilon_n(\lambda)| \le \frac{2}{\lambda^{m+1}}\sum_{s=0}^{n-1}|a_s|\frac{\Gamma[(s+1)/2]}{|\Gamma[(s+1-2m)/2]|}\rho^{-(2m-1-s)/2}. \tag{2.30}$$

The two estimates in (2.28) and (2.30) clearly establish the asymptotic nature of the expansion in (2.27). Taking just the first term, we have from (2.6) and (2.9)

$$I_2(\lambda) \sim g(c)\sqrt{\frac{\pi}{2f''(c)}}e^{i[\lambda f(c)+\pi/4]}. \tag{2.31}$$

The integral $I_1(\lambda)$ in (2.3) can be dealt with in exactly the same manner, after first changing the variable of integration from t to $-t$. The final result is that $I_1(\lambda)$ has the same asymptotic behavior as $I_2(\lambda)$ given in (2.31). The stationary-phase formula (2.1) now follows from (2.3).

3. Coalescing Stationary Points

We now consider a case in which the phase function $f(x)$ in the integral involves an auxiliary parameter α, say $f(t) = f(t, \alpha)$. We assume that $f(t, \alpha)$ has two local extrema in (a, b), say a maximum at $t = t_+(\alpha)$ and a minimum at $t = t_-(\alpha)$, and that these two extrema approach each other at $t = \bar{t} \in (a, b)$ when α tends to a critical value, say $\alpha = \alpha_0$. Thus,

$$f'(t_+, \alpha) = f'(t_-, \alpha) = 0, \qquad f''(t_\pm, \alpha) \ne 0 \tag{3.1}$$

for $\alpha \neq \alpha_0$, and

$$f'(\bar{t}, \alpha_0) = f''(\bar{t}, \alpha_0) = 0, \qquad f'''(\bar{t}, \alpha_0) \neq 0. \tag{3.2}$$

According to the approximation formulas in (2.1) and (2.2), the asymptotic behavior of $I(\lambda) = I(\lambda, \alpha)$ for large λ is given by

$$\begin{aligned} I(\lambda, \alpha) \sim\ & g(t_-)\sqrt{\frac{2\pi}{\lambda f''(t_-)}}e^{i[\lambda f(t_-)+\pi/4]} \\ & + g(t_+)\sqrt{\frac{2\pi}{-\lambda f''(t_+)}}e^{i[\lambda f(t_+)-\pi/4]}. \end{aligned} \tag{3.3}$$

In view of (3.2), the two terms on the right-hand side will blow up when α approaches α_0. Our aim here is to show that there is an asymptotic formula which holds uniformly for α in a neighborhood of α_0. Since the simplest function which exhibits the situation described in (3.1) – (3.2) is a cubic polynomial, Chester, Friedman and Ursell [3] introduced in their seminal paper the cubic transformation

$$f(t, \alpha) = \frac{1}{3}u^3 - \zeta u + \eta. \tag{3.4}$$

Furthermore, they showed that when $f(t, \alpha)$ is an analytic function of t and α, and when the coefficients ζ and η are chosen so that

$$\zeta^{3/2} = \frac{3}{4}[f(t_-, \alpha) - f(t_+, \alpha)], \tag{3.5}$$

$$\eta = \frac{1}{2}[f(t_-, \alpha) + f(t_+, \alpha)], \tag{3.6}$$

the cubic equation (3.4) has a real-valued solution $u(t, \alpha)$ such that $\partial u/\partial t \neq 0$ at $(\bar{t}, \alpha_0)$. An extension of this result to C^∞ case can be found in the book of Hörmander [5, p. 204, Theorem 7.5.13].

From (2.27), it is readily seen that the contribution from an endpoint (in our case, it is $t = b$, i.e., $u = \rho$) is of order $O(\lambda^{-1})$, whereas the contribution from the stationary point (in our case, it is $t = c$, i.e., $u = 0$) is of order $O(\lambda^{-1/2})$. Thus, it is reasonable to expect that to the first approximation, the contribution from the end points can be neglected. We may therefore assume without loss of generality that the support of the amplitude function $g(t)$ in (1.1) is contained in the interval where the cubic transformation (3.4) holds and that it contains the two local extrema t_+ and t_-. Hence, upon changing the integration variable from t to u, we have

$$I(\lambda, \alpha) = \int_{-\infty}^{\infty} \varphi_0(u) e^{i\lambda(u^3/3 - \zeta u + \eta)} du, \tag{3.7}$$

where

$$\varphi_0(u) = g(t)\frac{dt}{du} \tag{3.8}$$

vanishes smoothly and identically outside a finite u-interval. An asymptotic approximation of $I(\lambda, \alpha)$ can be obtained as follows. We write

$$\varphi_0(u) = a_0 + b_0 u + (u^2 - \zeta)\psi_0(u), \tag{3.9}$$

where

$$a_0 = \frac{1}{2}[\varphi_0(\zeta^{1/2}) + \varphi_0(-\zeta^{1/2})] \tag{3.10}$$

and

$$b_0 = \frac{1}{2\zeta^{1/2}}[\varphi_0(\zeta^{1/2}) - \varphi_0(-\zeta^{1/2})]; \tag{3.11}$$

see Wong [14, p. 368]. Inserting (3.9) in (3.7) yields

$$\begin{aligned} I(\lambda, \alpha) = 2\pi e^{i\lambda\eta}\Bigg[&\mathrm{Ai}(-\lambda^{2/3}\zeta)\frac{a_0}{\lambda^{1/3}} \\ &- i\mathrm{Ai}'(-\lambda^{2/3}\zeta)\frac{b_0}{\lambda^{2/3}} + \varepsilon_1(\lambda, \alpha)\Bigg], \end{aligned} \tag{3.12}$$

where

$$\varepsilon_1(\lambda, \alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(u^2 - \zeta)\psi_0(u)e^{i\lambda(u^3/3 - \zeta u)}du. \tag{3.13}$$

In (3.12), we have made use of the integral representation

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{i\left(\frac{1}{3}t^3 + xt\right)\right\}dt. \tag{3.14}$$

From (3.10) and (3.11), it is clear that

$$\lim_{\sqrt{\zeta}\to 0} a_0 = \varphi_0(0) \qquad \text{and} \qquad \lim_{\sqrt{\zeta}\to 0} b_0 = \varphi_0'(0). \tag{3.15}$$

With a_0 and b_0 so chosen, it is also easily seen that the function

$$\psi_0(u) = \frac{\varphi_0(u) - a_0 - b_0 u}{u^2 - \zeta} \tag{3.16}$$

is continuously differentiable in $(-\infty, \infty)$. To the integral in (3.13), we now apply an integration by parts and the result is

$$\varepsilon_1(\lambda, \alpha) = \frac{i}{2\pi\lambda}\int_{-\infty}^{\infty}\psi_0'(u)e^{i\lambda(u^3/3 - \zeta u)}du. \tag{3.17}$$

Since $\varphi_0(u)$ vanishes identically outside a finite interval, it follows from (3.16) that $\psi_0(u) = O(u^{-1})$ and $\psi_0'(u) = O(u^{-2})$ as $u \to \pm\infty$. Therefore we have

$$|\varepsilon_1(\lambda, \alpha)| \le \frac{1}{2\pi\lambda}\int_{-\infty}^{\infty}|\psi_0'(u)|du = O(\lambda^{-1}), \tag{3.18}$$

where the O-term holds uniformly for α in a neighborhood of α_0. Since both $\sqrt{\zeta}$ and $-\sqrt{\zeta}$ are real stationary points in (3.4), ζ is always non-negative, a fact which also follows from (3.5). In view of the well known results

$$\mathrm{Ai}(-x) \sim \frac{1}{\pi^{1/2}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{1}{4}\pi\right)$$

and

$$\mathrm{Ai}'(-x) \sim \frac{x^{1/4}}{\pi^{1/2}} \sin\left(\frac{2}{3}x^{3/2} - \frac{1}{4}\pi\right),$$

it is evident that the error term $\varepsilon_1(\lambda, \alpha)$ is indeed smaller than the first two terms in (3.12). Furthermore, the second term is of the same asymptotic order as the first term when $\zeta \neq 0$. Therefore, both terms must be kept in the first approximation to $I(\lambda, \alpha)$. Finally, we note that the integral in (3.17) is exactly of the same form as that in (3.7), except that $\varphi_0(u)$ is now replaced by $\varphi_1(u) := \psi_0'(u)$. Thus, the above procedure can be repeated to produce an infinite compound asymptotic expansion. The material in this section is taken from [7, Sec. 3].

4. Two-dimensional Approximation

A heuristic argument for (1.4) proceeds as follows. Since (x_0, y_0) is a stationary point, the Taylor expansion of $f(x, y)$ at (x_0, y_0) is given by

$$\begin{aligned} f(x,y) = f(x_0,y_0) &+ \frac{1}{2}\alpha(x-x_0)^2 + \frac{1}{2}\beta(y-y_0)^2 \\ &+ \gamma(x-x_0)(y-y_0) + \cdots, \end{aligned} \tag{4.1}$$

where $\alpha = f_{xx}(x_0, y_0), \beta = f_{yy}(x_0, y_0)$ and $\gamma = f_{xy}(x_0, y_0)$. Let $\xi = (x - x_0)$ and $\eta = (y - y_0)$. Then approximately, we have

$$f(x,y) \approx f(x_0,y_0) + \frac{1}{2}\alpha\xi^2 + \frac{1}{2}\beta\eta^2 + \gamma\xi\eta. \tag{4.2}$$

Substituting (4.2) in (1.3), and replacing $g(x, y)$ by $g(x_0, y_0)$, we obtain

$$\begin{aligned} I(\lambda) \approx\ & g(x_0,y_0)e^{i\lambda f(x_0,y_0)} \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\lambda(\alpha\xi^2+2\gamma\xi\eta+\beta\eta^2)} d\xi d\eta. \end{aligned} \tag{4.3}$$

The double integral on the right-hand side can be evaluated in closed form, and this gives

$$I(\lambda) \approx \frac{2\pi i\sigma}{\lambda\sqrt{|\alpha\beta-\gamma^2|}} g(x_0,y_0)e^{i\lambda f(x_0,y_0)}, \tag{4.4}$$

where $\sigma = 1$ if $\alpha\beta > \gamma^2$ and $\alpha > 0; \sigma = -1$ if $\alpha\beta > \gamma^2$ and $\alpha < 0$; and $\sigma = -i$ if $\alpha\beta < \gamma^2$, thus proving (1.4). The quantity $\kappa = |\alpha\beta - \gamma^2|$ is the Gaussian curvature of the surface $z = f(x, y)$ at (x_0, y_0).

To give a rigorous derivation of (4.4), we first assume that the stationary point (x_0, y_0) occurs at the origin $(0,0)$, and that the origin is an interior point of integration domain D in (1.3). Next, we point out that the cross-product term $(x - x_0)(y - y_0)$ in (4.1) can be eliminated by a linear transformation. Finally, we note that since the constant term $f(x_0, y_0)$ in (4.1) only contributes a factor $\exp\{i\lambda f(x_0, y_0)\}$ to the double integral $I(\lambda)$ in (1.3), we can take $f(x_0, y_0) = 0$ without loss of generality. The Taylor expansion (4.1) now reduces to the form

$$f(x, y) = f_{20}\, x^2 + f_{02}\, y^2 + \cdots , \tag{4.5}$$

which can be written as

$$f(x, y) = f_{20}\, x^2[1 + P(x, y)] + f_{02}\, y^2[1 + Q(x, y)], \tag{4.6}$$

where P and Q are power series in x and y satisfying $P(0,0) = Q(0,0) = 0$. In the integral in (1.3), we make the change of variables

$$u = x[1 + P(x, y)]^{1/2}, \qquad v = y[1 + Q(x, y)]^{1/2} \tag{4.7}$$

so that $f(x, y) = f_{20}\, u^2 + f_{02}\, v^2$. Let D' denote the image of D under this change of variables, and put

$$G(u, v) = g(x, y)\frac{\partial(x, y)}{\partial(u, v)} \qquad \text{and} \qquad F(u, v) = f_{20}\, u^2 + f_{02}\, v^2. \tag{4.8}$$

The double integral $I(\lambda)$ then becomes

$$I(\lambda) = \iint_{D'} G(u, v) e^{i\lambda F(u,v)} du dv. \tag{4.9}$$

The new amplitude function $G(u, v)$ has the Maclaurin expansion

$$G(u, v) = \sum G_{ij} u^i v^j, \qquad G_{00} = g(0, 0), \tag{4.10}$$

which is obtained by inverting the variables in (4.7). The coefficients G_{ij} in (4.10) can be expressed in terms of the derivatives of f and g at $(0,0)$. Let m and M denote the infimum and supremum of F in D' (or equivalently, of f in D), respectively. By the method of resolution of multiple integrals [4, pp. 445-455], the double integral $I(\lambda)$ in (4.9) can be written as

$$I(\lambda) = \int_m^M h(t) e^{i\lambda t} dt, \tag{4.11}$$

where

$$h(t) = \int_{F(u,v)=t} \frac{G(u, v)}{\sqrt{F_u^2 + F_v^2}} d\sigma, \tag{4.12}$$

σ being the arc length of the curve $F(u,v)=t$.

In this lecture, we consider only the case when both f_{20} and f_{02} are positive, and refer the reader to [14, Chapter VIII] for a detailed discussion of other cases. Thus, in the present case, $(0,0)$ is a local minimum and $m=0$ in (4.11). Since the contribution from an end-point is of lower order than that from a stationary point, as remarked earlier in Sec. 3, we may assume that the support of $h(t)$ in (4.11) lies in a small interval containing the origin $t=0$. This can be achieved by making the support of $g(x,y)$, and hence that of $G(u,v)$, sufficiently small. From (4.6), it is clear that

$$\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(0,0)}=1.$$

We assume that $\partial(x,y)/\partial(u,v)$ is positive in the support of G. To derive an asymptotic expansion for $h(t)$ as $t\to 0^+$, we use (4.12). The second equation in (4.8) shows that the level curves $F(u,v)=t$ are ellipses. Let

$$u=\left(\frac{\xi}{f_{20}}\right)^{1/2}\cos\eta \qquad \text{and} \qquad v=\left(\frac{\xi}{f_{02}}\right)^{1/2}\sin\eta. \tag{4.13}$$

Clearly $\xi=f_{20}\,u^2+f_{02}\,v^2=F(u,v)$ and

$$\frac{1}{|\nabla F|}d\sigma=\frac{1}{2\sqrt{f_{20}f_{02}}}d\eta.$$

Thus from (4.12) it follows that

$$h(t)=\int_{\xi=t}\Phi(\xi,\eta)d\eta=\int_0^{2\pi}\Phi(t,\eta)d\eta, \tag{4.14}$$

where

$$\Phi(\xi,\eta)=\frac{1}{2\sqrt{f_{20}f_{02}}}G(u,v).$$

The second equality in (4.14) holds when t is sufficiently small. Put

$$G^*_{ij}=\frac{G_{ij}}{f_{20}^{1/2}f_{02}^{1/2}}.$$

Then $\Phi(\xi,\eta)$ has the asymptotic representation

$$\Phi(\xi,\eta)\sim\frac{1}{2\sqrt{f_{20}f_{02}}}\sum G^*_{ij}\,\xi^{(i+j)/2}\cos^i\eta\sin^j\eta, \tag{4.15}$$

as $\xi\to 0^+$. Since

$$\int_0^{2\pi}\cos^i\eta\sin^j\eta d\eta=0 \qquad \text{for} \qquad i,j=1,3,5,\cdots$$

and

$$\int_0^{2\pi} \cos^{2m}\eta \sin^{2n}\eta d\eta = 4\int_0^{\pi/2} \cos^{2m}\eta \sin^{2n}\eta d\eta$$
$$= 2\frac{\Gamma\big(m+\frac{1}{2}\big)\Gamma\big(n+\frac{1}{2}\big)}{(m+n)!},$$

we find, on inserting (4.15) in (4.14), that

$$h(t) \sim \frac{1}{\sqrt{f_{20}f_{02}}} \sum \frac{\Gamma\big(m+\frac{1}{2}\big)\Gamma\big(n+\frac{1}{2}\big)}{(m+n)!} G^*_{2m,2n} t^{m+n}, \tag{4.16}$$

as $t \to 0^+$. Since $h(t)$ vanishes identitically outside a small neighborhood of the origin, there is no contribution from the endpoint $t = M$, and the final result

$$I(\lambda) \sim \frac{1}{\sqrt{f_{20}f_{02}}} \sum G^*_{2m,2n} e^{i\pi(m+n+1)/2} \frac{\Gamma\big(m+\frac{1}{2}\big)\Gamma\big(n+\frac{1}{2}\big)}{\lambda^{m+n+1}}, \tag{4.17}$$

as $\lambda \to +\infty$, now follows from (4.11) by termwise integration in the sense of Abel summability; see Sec. 2.

Taking into account the elimination of the cross product xy and the constant term $f(0,0)$ in the Maclaurin expansion of $f(x,y)$, the leading term in (4.17) agrees with the required result (1.4) when $\sigma_f = 2$.

5. A Uniform Asymptotic Formula

As a generalization of the uniform asymptotic approximation given in Sec. 3, we consider the double integral

$$I(\lambda,\alpha) = \iint_D g(x,y) e^{i\lambda f(x,y,\alpha)} dxdy, \tag{5.1}$$

where the phase function $f(x,y,\alpha)$ has two simple stationary points $(x_+(\alpha), y_+(\alpha))$ and $(x_-(\alpha), y_-(\alpha))$ in the integration domain D, which coalesce at an interior point (x_0, y_0) in D as α approaches a critical value α_0. Problems of this nature arise in propeller acoustics; see Chapman [2] and Prentice [10]. In fact, there is an entire book written by Borovikov [1] on this topic. The material in this section is taken from Qiu and Wong [11].

We shall write $f_\alpha(x,y) \equiv f(x,y,\alpha)$. Without loss of generality, we may take $\alpha_0 = 0, (x_0,y_0) = (0,0)$ and $f(0,0,\alpha) = 0$ for all small α. Under these assumptions, it is readily seen that $(0,0)$ is a degenerate stationary point of $f_0(x,y) = f(x,y,0)$. If we put

$$a_{ij} \equiv \left.\frac{\partial^{i+j} f_0}{\partial x^i \partial y^j}\right|_{(0,0)}, \tag{5.2}$$

then the gradient of f_0 at $(0,0)$ is

$$\nabla f_0(0,0) = (a_{10}, a_{01}) = (0,0) \tag{5.3}$$

and the Hessian of f_0 at $(0,0)$ is

$$\det f_0''(0,0) = a_{20}a_{02} - a_{11}^2 = 0. \tag{5.4}$$

The Taylor expansion of $f_0(x,y)$ at the origin is given by

$$\begin{aligned} f_0(x,y) = {} & \frac{1}{2}(a_{20}\, x^2 + 2a_{11}xy + a_{02}\, y^2) \\ & + \frac{1}{6}(a_{30}\, x^3 + 3a_{21}\, x^2y + 3a_{12}\, xy^2 + a_{03}\, y^3) + \cdots . \end{aligned} \tag{5.5}$$

We now recall the following result from catastrophe theory, known as the splitting lemma [9, p.95].

LEMMA. *Let $f(x,y,\alpha)$ be a smooth function of (x,y,α) defined in a neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R}$. Write $f_\alpha(x,y) = f(x,y,\alpha)$. Suppose that the Taylor expansion of $f_0(x,y)$ at $(0,0)$ is of the form (5.5), and that condition (5.4) holds with a_{20}, a_{11} and a_{02} not all zero. Then, for all small values of α, there exists a smooth and one-to-one mapping $(x,y) \to (s,t)$ given by $s = s_\alpha(x,y)$ and $t = t_\alpha(x,y)$ with $s_0(0,0) = t_0(0,0) = 0$, which depends also smoothly on α, such that*

$$f_\alpha(x,y) = \varepsilon s^2 + g_\alpha(t), \qquad \varepsilon = \pm 1,$$

where $g_\alpha(t)$ is smooth in both t and α and $g_0(0) = 0$.

Using the splitting lemma above and the cubic transformation in (3.4), it can be established that if either

$$a_{03}\, a_{20}^2 - 3a_{12}\, a_{20}\, a_{11} + 3a_{21}\, a_{20}\, a_{02} - a_{30}\, a_{02}\, a_{11} \neq 0 \tag{5.6}$$

or

$$a_{30}\, a_{02}^2 - 3a_{21}\, a_{02}\, a_{11} + 3a_{12}\, a_{20}\, a_{02} - a_{03}\, a_{20}\, a_{11} \neq 0, \tag{5.7}$$

then for all small values of α, $f_\alpha(x,y)$ can be mapped into the canonical form

$$f_\alpha(x,y) = \varepsilon u^2 + \frac{1}{3}v^3 + \xi(\alpha)v + \eta(\alpha) \tag{5.8}$$

by a smooth and one-to-one mapping $(x,y) \to (u,v)$ given by $u = u_\alpha(x,y)$ and $v = v_\alpha(x,y)$, where $\varepsilon = \operatorname{sgn} a_{20}$ if $a_{20} \neq 0$ or $\varepsilon = \operatorname{sgn} a_{02}$ if $a_{02} \neq 0$, and where $\xi(\alpha)$ and $\eta(\alpha)$ are smooth functions with $\xi(0) = \eta(0) = 0$. Furthermore, $u_0(0,0) = v_0(0,0) = 0$. The functions $\xi(\alpha)$ and $\eta(\alpha)$ are given by

$$\xi(\alpha) = -\left[\frac{3}{4}\Big(f_\alpha\big(x_-(\alpha), y_-(\alpha)\big) - f_\alpha\big(x_+(\alpha), y_+(\alpha)\big)\Big)\right]^{2/3} \tag{5.9}$$

$$\eta(\alpha) = \frac{1}{2}\big[f_\alpha(x_+(\alpha), y_+(\alpha)) + f_\alpha(x_-(\alpha), y_-(\alpha))\big]. \tag{5.10}$$

Returning to (5.1), we assume as in Sec. 4 that the support of $g(x, y)$ is contained in D. This is reasonable at least to the leading order approximation, since the contribution from the boundary is always of lower asymptotic order than that from stationary points. We also assume that α is sufficiently small so that (5.8) holds inside the support of g. Making the change of variables from (x, y) to (u, v), we have

$$I(\lambda, \alpha) = e^{i\lambda\eta(\alpha)} \iint_{\mathbb{R}^2} \varphi_0(u, v, \alpha) e^{i\lambda(\varepsilon u^2 + v^3/3 + \xi(\alpha)v)} dv du, \tag{5.11}$$

where

$$\varphi_0(u, v, \alpha) = g(x, y, \alpha)\left|\frac{\partial(x, y)}{\partial(u, v)}\right|. \tag{5.12}$$

Note that $\varphi_0(u, v, \alpha)$ vanishes identically outside a neighborhood of the origin in $\mathbb{R}^2$. To obtain the values of $\varphi_0(u, v, \alpha)$ at the stationary points $(0, \pm(-\xi(\alpha))^{1/2})$, we need to evaluate the Jacobian $|\partial(x, y)/\partial(u, v)|$ at these points. The result is

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right|^2_{(0, \pm(-\xi(\alpha))^{1/2})} = \frac{\pm 4\varepsilon(-\xi(\alpha))^{1/2}}{H_{f_\alpha}(x_\pm(\alpha), y_\pm(\alpha))}, \tag{5.13}$$

where $H_f(x, y)$ denotes the Hessian of the function f at (x, y).

Analogous to (3.9) in the one-dimensional case, we now write

$$\begin{aligned} \varphi_0(u, v, \alpha) = a_0(\alpha) + b_0(\alpha)v \\ + (v^2 + \xi(\alpha))\psi_{02}(v, \alpha) + 2u\varepsilon\psi_{01}(u, v, \alpha). \end{aligned} \tag{5.14}$$

Substitute (5.14) in (5.11) and carry out integration term-by-term. Instead of using integration by parts as in Sec. 3, here we need Green's theorem. The two-term asymptotic expansion for the integral (5.11) is given by

$$\begin{aligned} I(\lambda, \alpha) \sim e^{i(\lambda\eta(\alpha) + \varepsilon\pi/4)} \frac{2\pi^{3/2}}{\lambda^{5/6}} \\ \times \left[a_0(\alpha)\mathrm{Ai}(\lambda^{2/3}\xi) + \frac{b_0(\alpha)}{i\lambda^{1/3}}\mathrm{Ai}'(\lambda^{2/3}\xi)\right]. \end{aligned} \tag{5.15}$$

To obtain the coefficients $a_0(\xi)$ and $b_0(\xi)$, we put $u = 0$ and $v = \pm(-\xi(\alpha))^{1/2}$ in (5.14). This gives

$$a_0(\alpha) = \frac{1}{2}\big[\varphi_0(0, (-\xi(\alpha))^{1/2}, \alpha) + \varphi_0(0, -(-\xi(\alpha))^{1/2}, \alpha)\big] \tag{5.16}$$

and

$$\begin{aligned} b_0(\alpha) = \frac{1}{2(-\xi(\alpha))^{1/2}}\big[\varphi_0(0, (-\xi(\alpha))^{1/2}, \alpha) \\ - \varphi_0(0, -(-\xi(\alpha))^{1/2}, \alpha)\big]. \end{aligned} \tag{5.17}$$

A combination of (5.12), (5.13), (5.16) and (5.17) yields

$$a_0(\alpha) = (-\xi)^{1/4}\left[\frac{g(x_+, y_+, \alpha)}{|H_{f\alpha}(x_+, y_+)|^{1/2}} + \frac{g(x_-, y_-, \alpha)}{|H_{f\alpha}(x_-, y_-)|^{1/2}}\right]$$

and

$$b_0(\alpha) = (-\xi)^{1/4}\left[\frac{g(x_+, y_+, \alpha)}{|H_{f\alpha}(x_+, y_+)|^{1/2}} - \frac{g(x_-, y_-, \alpha)}{|H_{f\alpha}(x_-, y_-)|^{1/2}}\right].$$

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