On a Uniform Treatment of Darboux's Method

by

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Abstract

Let $F(z)$ be an analytic function in $|z| < 1$. If $F(z)$ has only a finite number of algebraic singularities on the unit circle $|z| = 1$, then Darboux's method can be used to give an asymptotic expansion for the coefficient of z^n in the Maclaurin expansion of $F(z)$. However, the validity of this expansion ceases to hold, when the singularities are allowed to approach each other. A special case of this confluence was studied by Fields in 1968. His results have been considered to be too complicated by others, and desires have been expressed to investigate whether any simplification is feasible. In this paper, we shall show that simplification is indeed possible. In the case of two coalescing algebraic singularities, our expansion involves only two Bessel functions of the first kind.

Key words. Darboux's method, uniform asymptotic expansion, coalescing algebraic singularities.

AMS subject classifications. 41A60, 33C10, 33E20.

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1 Introduction

The problem of obtaining the asymptotic behavior for the coefficients a_n of the Maclaurin expansion

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

arises in many instances. The main sources for these problems are number theory and combinatorics. Darboux was the first to attack general problems of this nature. When $F(z)$ has only a finite number of singularities on its circle of convergence, all of which are algebraic in nature, he showed that an asymptotic expansion could be obtained for a_n as $n \to \infty$. The method he used is now known as Darboux's method; see, e.g., Wong [13, pp. 116-122].

Darboux's method dates back to 1878, and although often used it does not appear to have been extended until around 1970. In a 1974 paper, Wong & Wyman [14] have given a generalization of Darboux's method which allows the generating function $F(z)$ in (1.1) to have logarithmic-type singularities on its circle of convergence. Like Darboux, they dealt with problems in which the locations of the singularities are fixed.

When the singularities are free to move on the circle of convergence, Darboux's method will continue to work only if their essential configuration remains the same as the relative positions vary. This method breaks down when two or more singularities coalesce with each other. In that case, the asymptotic expansion will involve transcendental functions instead of elementary ones.

In 1968, Fields [6] made a uniform treatment of Darboux's method when two or three singularities coalesce. More precisely, he considered the case in which

$$F(z,\theta) = (1-z)^{-\lambda}\left[(e^{i\theta}-z)(e^{-i\theta}-z)\right]^{-\Delta} f(z,\theta) = \sum_{n=0}^{\infty} a_n(\theta) z^n, \tag{1.2}$$

where the Maclaurin expansion converges for $|z| < 1$, λ and Δ are bounded quantities, the branches of $(1-z)^{-\lambda}$ and $\left[(e^{i\theta}-z)(e^{-i\theta}-z)\right]^{-\Delta}$ are chosen such that each of them reduces to 1 at $z = 0$, and $f(z,\theta)$ is analytic in $|z| \le e^{\eta}$ ($\eta > 0$), uniformly for $\theta \in [0,\pi]$. In [6], Fields first expressed $a_n(\theta)$ as a Cauchy integral, then made a change of variable and a rescaling, and finally obtained a generalized asymptotic expansion in the sense of Erdélyi & Wyman [5]. His results are uniform in certain θ-intervals depending on n. Generalized asymptotic expansions, extending Poincaré's original definition, are commonly used in the study of uniform asymptotics; see, e.g., Chester, Friedman & Ursell [3], Bleistein [1], Frenzen & Wong [7], and also Wong [13, Ch. VII].

Despite the fact that Fields' results have achieved the so-called *uniform reduction* in the sense of Olver [11, p. 102], they are found to be too complicated for any practical application; see, e.g., Erdélyi [4, p. 167], Olver [11, pp. 112-113] and Wong [13, p.145]. The following remark, made by Olver, is typical: *it may be desirable to investigate* *whether any simplifications are feasible since the results in [6] are rather complicated to apply in their present form.* The purpose of the present paper is just to pursue such an investigation; i.e., to derive simpler forms of uniform asymptotic expansions when two or more algebraic singularities, on the circle of convergence, coalesce with each other as some parameter approaches a certain critical value.

As a point of information, let us mention two relevant pieces of work. One is that of Bleistein [1], the main concern of which is uniform asymptotic expansion of integrals with many nearby saddle points and algebraic singularities. At first impression, one may even think that the problem under present investigation is a special case of that treated in Bleistein [1]. This is certainly not the case, and furthermore, Bleistein's result is incomplete, as pointed out by Olver [11, p. 110], Ursell [12] and especially Erdélyi [4, p. 155]. The other related work is given in [15] concerning Jacobi polynomials. In fact, it is this latter work that has motivated us to carry out the current research, although the method used in that paper cannot be extended to the present case. Since the present paper deals with the asymptotics of late coefficients, it is probably also appropriate to mention some more recent work in this direction. For instance, in [2] Berry and Howls have studied the asymptotics of late coefficients that arise in uniform asymptotic expansions of integrals with coalescing saddles. A rigorous version of their results can be found in Olde Daalhuis [8].

In this paper, we shall concentrate on the derivation of uniform asymptotic expansions in the situations described above. In a sense, our method is similar to those used in Chester, Friedman & Ursell [3], Bleistein [1], and Frenzen & Wong [7]. However, a technique suggested by Olde Daalhuis & Temme [9] will play a central role in establishing the error estimates. As a consequence, simple uniform asymptotic expansions and error bounds are obtained. The main results are stated in Theorems 1 and 2; the coefficients and the remainder term in each of the uniform expansions are presented recursively and hence can be calculated successively.

The arrangement of the present paper is as follows. In Sections 2 and 3, we consider in detail a simple yet typical case, namely, two algebraic singularities on the circle of convergence coalescing with each other when a parameter approaches a critical value. This is essentially the situation in (1.2) with $\theta \in [0, \pi - \delta]$, where $\delta > 0$. The derivation of the uniform asymptotic expansion is carried out in Section 2, while the rigorous proof of the boundedness of the coefficients and the construction of error bounds are presented in Section 3. In Section 4, we provide a brief discussion of the general case when many algebraic singularities coalesce with each other at $z = 1$ as the parameter θ tends to 0. Specific examples are given in the final section, where some possible extensions are also mentioned.

2 Two points: derivation

In this and the next section, we will concentrate on a special case

$$F(z,\theta) = \left[(e^{i\theta} - z)(e^{-i\theta} - z)\right]^{-\alpha} f(z,\theta) = \sum_{n=0}^{\infty} a_n(\theta) z^n, \tag{2.1}$$

where $f(z,\theta)$ is a function analytic in $|z| \le e^{\eta}$ with $\eta > 0$. The uniformity in the asymptotic behavior of $a_n(\theta)$, as $n \to \infty$, is in $\theta \in [0, \pi - \delta]$. In (2.1), α is not an integer.

Using Cauchy formula, we have from (2.1)

$$a_n(\theta) = \frac{1}{2\pi i} \int_C f(z,\theta) \left(1 - 2z\cos\theta + z^2\right)^{-\alpha} \frac{dz}{z^{n+1}}, \tag{2.2}$$

where C is a simple closed contour which encloses $z = 0$ but not $z = e^{\pm i\theta}$ and lies in the domain of z-analyticity of $f(z,\theta)$. We may choose C so that it consists of two portions C_I and C_E, where C_I is a curve starting from $z = e^{-0i}e^{\eta}$, enclosing $z = e^{\pm i\theta}$ but not $z = 0$ in clockwise orientation, and ending at $z = e^{0i}e^{\eta}$, while C_E is the circle $|z| = e^{\eta}$, oriented counterclockwisely; see Figure 1.

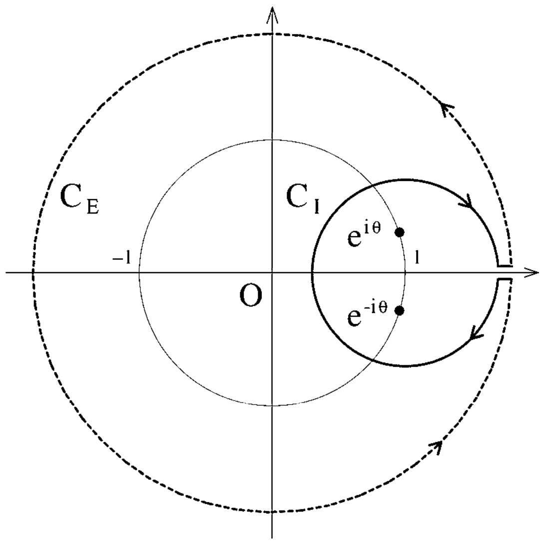

Figure 1. The contour in (2.2)

As we shall see, the contribution from C_E is exponentially small. Indeed, let us define

$$\mathcal{A}_n(\theta) = \frac{1}{2\pi i} \int_{C_I} f(z,\theta) \left(1 - 2z\cos\theta + z^2\right)^{-\alpha} \frac{dz}{z^{n+1}} \tag{2.3}$$

and

$$\varepsilon_E(\theta) = \frac{1}{2\pi i} \int_{C_E} f(z,\theta) \left(1 - 2z\cos\theta + z^2\right)^{-\alpha} \frac{dz}{z^{n+1}}. \tag{2.4}$$

Along C_E, we have

$$(e^\eta - 1)^2 \le \left|1 - 2z\cos\theta + z^2\right| \le (e^\eta + 1)^2$$

and hence

$$|\varepsilon_E(\theta)| = \left|\frac{1}{2\pi i}\int_{C_E} f(z,\theta)\left(1 - 2z\cos\theta + z^2\right)^{-\alpha}\frac{dz}{z^{n+1}}\right| \le c\,(f,\eta)e^{-\eta n}, \tag{2.5}$$

where $c\,(f,\eta)$ is a positive constant. In fact, one may choose

$$c\,(f,\eta) = \max_{|z|=e^\eta}\{\,|f(z,\theta)|\,\}\cdot\max\left\{(e^\eta - 1)^{-2\alpha},(e^\eta + 1)^{-2\alpha}\right\}.$$

From (2.2) – (2.5), it follows that

$$a_n(\theta) = \mathcal{A}_n(\theta) + \varepsilon_E(\theta), \tag{2.6}$$

where $|\varepsilon_E| \le c\,(f,\eta)e^{-\eta n}$.

Now we consider the behavior of $\mathcal{A}_n(\theta)$. Making the change of variable

$$z = e^{-\theta s} \tag{2.7}$$

in (2.3) yields

$$\mathcal{A}_n(\theta) = \frac{\theta^{1-2\alpha}}{2\pi i}\int_\Gamma h_0(s,\theta)(s^2+1)^{-\alpha}e^{n\theta s}ds, \tag{2.8}$$

where

$$h_0(s,\theta) = f(e^{-\theta s},\theta)\left[\left(\frac{e^{-s\theta} - e^{i\theta}}{(-s-i)\theta}\right)\left(\frac{e^{-s\theta} - e^{-i\theta}}{(-s+i)\theta}\right)\right]^{-\alpha} \tag{2.9}$$

is analytic in s for $\mathrm{Re}\,s \ge -\eta/\theta$ and $|s \pm i| < 2\pi/\theta$. (The last condition can, in fact, be replaced by $|\mathrm{Im}\,s| < \frac{2\pi}{\theta} - 1$.) In (2.8), Γ is the image of C_I under transformation (2.7). That is, Γ is the counterclockwisely oriented curve in the s-plane which starts at $e^{-i\pi}\eta/\theta$, ends at $e^{i\pi}\eta/\theta$, and encloses both $s = \pm i$.

We further introduce the notations

$$T_1(x) := \frac{1}{2\pi i}\int_{\Gamma_0}(s^2+1)^{-\alpha}e^{xs}ds\ ,\ T_2(x) := \frac{1}{2\pi i}\int_{\Gamma_0}s(s^2+1)^{-\alpha}e^{xs}ds, \tag{2.10}$$

where Γ_0 is a Hankel-type loop which starts and ends at $-\infty$, and encircles $s = \pm i$ in the positive sense. It is readily verified that $\frac{d}{dx}T_1(x) = T_2(x)$.

To pick up the first level contribution from the integral in (2.8), we write

$$h_0(s,\theta) = \alpha_0(\theta) + s\beta_0(\theta) + (s^2+1)g_0(s,\theta), \tag{2.11}$$

where the coefficients $\alpha_0(\theta)$ and $\beta_0(\theta)$ are determined by setting $s = \pm i$. More precisely, we have

$$\alpha_0(\theta) = \frac{1}{2}(h_0(i,\theta) + h_0(-i,\theta))\ ,\ \beta_0(\theta) = \frac{1}{2i}(h_0(i,\theta) - h_0(-i,\theta)). \tag{2.12}$$

Note that $g_0(s,\theta)$ in (2.11) has the same domain of s-analyticity as $h_0(s,\theta)$. Inserting (2.11) into (2.8) and integrating the last term by parts yield

$$\mathcal{A}_n(\theta) = \theta^{1-2\alpha}\alpha_0(\theta)\left(T_1(n\theta) - \varepsilon_{T_1}\right) + \theta^{1-2\alpha}\beta_0(\theta)\left(T_2(n\theta) - \varepsilon_{T_2}\right) + \frac{1}{n}\varepsilon_1, \tag{2.13}$$

where

$$\varepsilon_{T_l} = \int_{e^{-i\pi}\infty}^{e^{-i\pi}\eta/\theta} s^{l-1}(s^2+1)^{-\alpha}e^{n\theta s}ds + \int_{e^{i\pi}\eta/\theta}^{e^{i\pi}\infty} s^{l-1}(s^2+1)^{-\alpha}e^{n\theta s}ds, \tag{2.14}$$

$l = 1, 2,$ and

$$\varepsilon_1 = \Sigma_1 + \varepsilon_{1,E}. \tag{2.15}$$

In (2.15),

$$\varepsilon_{1,E} = \theta^{-2\alpha}\cdot\frac{1}{2\pi i}\left[g_0(s,\theta)(s^2+1)^{1-\alpha}e^{n\theta s}\right]\Bigg|_{s=e^{-i\pi}\eta/\theta}^{s=e^{i\pi}\eta/\theta} \tag{2.16}$$

represents the end-point contribution and

$$\Sigma_1 = \frac{\theta^{1-2\alpha}}{2\pi i}\int_\Gamma h_1(s,\theta)(s^2+1)^{-\alpha}e^{n\theta s}ds, \tag{2.17}$$

where

$$\begin{aligned} h_1(s,\theta) &= -\frac{1}{\theta}(s^2+1)^{\alpha}\frac{d}{ds}\left[g_0(s,\theta)(s^2+1)^{1-\alpha}\right] \\ &= -\frac{1}{\theta}\left[(s^2+1)\frac{d}{ds} + 2(1-\alpha)s\right]g_0(s,\theta). \end{aligned} \tag{2.18}$$

It can be seen from (2.18) that $h_1(s,\theta)$ has the same domain of s-analyticity as $g_0(s,\theta)$, and hence as $h_0(s,\theta)$. It can also be seen that the integral representation (2.17) for Σ_1 is of the same form as (2.8) for $\mathcal{A}_n(\theta)$. Thus, the procedure can be repeated.

Define inductively

$$h_k(s,\theta) = \alpha_k(\theta) + s\beta_k(\theta) + \left(s^2+1\right)g_k(s,\theta), \qquad k = 0,1,2,\cdots, \tag{2.19}$$

and

$$h_{k+1}(s,\theta) = -\frac{1}{\theta}\left[(s^2+1)\frac{d}{ds} + 2(1-\alpha)s\right]g_k(s,\theta), \qquad k = 0,1,2,\cdots. \tag{2.20}$$

Repeated application of integration by parts as above gives the formal expansion

$$\begin{aligned} a_n(\theta) = \;& \theta^{1-2\alpha}T_1(n\theta)\sum_{k=0}^{m-1}\frac{\alpha_k(\theta)}{n^k} + \theta^{1-2\alpha}T_2(n\theta)\sum_{k=0}^{m-1}\frac{\beta_k(\theta)}{n^k} \\ & + \left\{\varepsilon_E + \sum_{k=1}^{m}\frac{\varepsilon_{k,E}}{n^k} - \theta^{1-2\alpha}\sum_{k=0}^{m-1}\frac{\alpha_k(\theta)\varepsilon_{T_1} + \beta_k(\theta)\varepsilon_{T_2}}{n^k} + \frac{1}{n^m}\Sigma_m\right\} \end{aligned} \tag{2.21}$$

for $m = 1, 2, \cdots$, where

$$\varepsilon_{k,E} = \theta^{-2\alpha} \cdot \frac{1}{2\pi i} \left[g_{k-1}(s,\theta)(s^2+1)^{1-\alpha} e^{ns\theta} \right] \Bigg|_{s=e^{-i\pi}\eta/\theta}^{s=e^{i\pi}\eta/\theta}, \quad k = 1, 2, \cdots, \tag{2.22}$$

and

$$\Sigma_m = \frac{\theta^{1-2\alpha}}{2\pi i} \int_\Gamma h_m(s,\theta)(s^2+1)^{-\alpha} e^{n\theta s} ds, \ m = 1, 2, \cdots. \tag{2.23}$$

One can see from (2.19) and (2.20) that $h_k(s,\theta)$ and $g_k(s,\theta)$ have the same domain of s-analyticity as $h_0(s,\theta)$.

To show that ε_{T_1} and ε_{T_2} are exponentially small, we set

$$I = \int_{\eta/\theta}^{\infty} (s^2+1)^{-\alpha} e^{-n\theta s} ds, \tag{2.24}$$

and make the change of variable $s = (t+1)\eta/\theta$. The integral in (2.24) becomes

$$I = \eta^{1-2\alpha} \theta^{2\alpha-1} e^{-\eta n} \int_0^\infty \left[(t+1)^2 + \frac{\theta^2}{\eta^2} \right]^{-\alpha} e^{-\eta n t} dt. \tag{2.25}$$

Note that $\theta^2/\eta^2 \geq 0$, and

$$\left[(t+1)^2 + \frac{\theta^2}{\eta^2} \right]^{-\alpha} \leq (t+1)^{-2\alpha}$$

for $\theta \in [0, \pi]$ and $t \geq 0$. Hence

$$|I| \leq C(\eta) \theta^{2\alpha-1} e^{-\eta n} \int_0^\infty (t+1)^{-2\alpha} e^{-\eta n t} dt \leq C(\eta) \theta^{2\alpha-1} \frac{1}{n} e^{-\eta n}, \tag{2.26}$$

where we have used $C(\eta)$ as a generic symbol to denote positive constants, independent of both θ and n, whose values may differ in different places. From (2.14) and (2.26), it follows that

$$\theta^{1-2\alpha} |\varepsilon_{T_1}| \leq C(\eta) \frac{1}{n} e^{-\eta n} \tag{2.27}$$

and

$$\theta^{2-2\alpha} |\varepsilon_{T_2}| \leq C(\eta) e^{-\eta n} \tag{2.28}$$

for $\theta \in [0, \pi]$. The last inequality is obtained by combining (2.14) with (2.26) and integrating by parts once in both integrals in (2.14).

It can also be shown that

$$T_1(n\theta) = \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(2\alpha)} (2n\theta)^{\alpha - \frac{1}{2}} J_{\alpha - \frac{1}{2}}(n\theta) = \frac{\sqrt{\pi}}{\Gamma(\alpha)} \left(\frac{n\theta}{2} \right)^{\alpha - \frac{1}{2}} J_{\alpha - \frac{1}{2}}(n\theta), \tag{2.29}$$

where $J_\nu(\tau)$ is the Bessel function of the first kind. The first equality is obtained by deforming Γ_0 so that $|s| > 1$ for $s \in \Gamma_0$, and expanding the factor $(s^2+1)^{-\alpha}$ in powers of $1/s$; (2.29) then follows from the series representation of the Bessel function; see, e.g. Wong [13, p. 231]. Similarly, we have

$$T_2(n\theta) = \frac{(\alpha-\frac{1}{2})\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{n\theta}{2}\right)^{\alpha-\frac{3}{2}} J_{\alpha-\frac{1}{2}}(n\theta) - \frac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{n\theta}{2}\right)^{\alpha-\frac{1}{2}} J_{\alpha+\frac{1}{2}}(n\theta) \tag{2.30}$$

or, equivalently,

$$T_2(n\theta) = \frac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{n\theta}{2}\right)^{\alpha-\frac{1}{2}} J_{\alpha-\frac{3}{2}}(n\theta). \tag{2.31}$$

3 Asymptotic nature of the expansion (2.21)

3.1. Statement of a main result. In the previous section, we have shown that the Maclaurin coefficients $a_n(\theta)$ in (2.1) can be expressed in terms of the integrals $T_1(n\theta)$ and $T_2(n\theta)$ as in (2.21), and that T_1 and T_2 can in turn be expressed in terms of the Bessel functions $J_{\alpha-\frac{1}{2}}$ and $J_{\alpha+\frac{1}{2}}$; cf. (2.29) and (2.30). In this section, we proceed to prove one of the main results of the present paper, namely, a uniform asymptotic expansion for $a_n(\theta)$ as n tends to infinity.

THEOREM 1. *Assume that $f(z,\theta)$ in (2.1) is uniformly bounded for $\theta \in [0,\pi]$, and is z-analytic in $|z| \le e^{\eta} (\eta > 0)$. For any integer $m \ge 1$, we have*

$$a_n(\theta) = \theta^{1-2\alpha} T_1(n\theta) \sum_{k=0}^{m-1} \frac{\alpha_k(\theta)}{n^k} + \theta^{1-2\alpha} T_2(n\theta) \sum_{k=0}^{m-1} \frac{\beta_k(\theta)}{n^k} + \varepsilon(\theta, m), \tag{3.1}$$

where

$$|\alpha_k(\theta)| \le M_k \tag{3.2}$$

and

$$|\beta_k(\theta)/\theta| \le M_k \tag{3.3}$$

for $k = 0, 1, 2, \cdots$, and

$$|\varepsilon(\theta, m)| \le M_m \frac{\theta^{1-2\alpha}}{n^m} [|T_1(n\theta)| + |T_2(n\theta)|] \tag{3.4}$$

for $m = 1, 2, 3, \cdots$. The positive constants $M_k, k = 0, 1, 2, \cdots$, are independent of θ for $\theta \in [0, \pi - \delta]$, $\delta > 0$, the coefficients $\alpha_k(\theta)$ and $\beta_k(\theta)$ are defined successively by (2.9), (2.19) and (2.20), and the remainder $\varepsilon(\theta, m)$ satisfies

$$\varepsilon(\theta, m) = \varepsilon_E + \sum_{k=1}^{m} \frac{\varepsilon_{k,E}}{n^k} - \theta^{1-2\alpha} \sum_{k=0}^{m-1} \frac{\alpha_k(\theta)\varepsilon_{T_1} + \beta_k(\theta)\varepsilon_{T_2}}{n^k} + \frac{1}{n^m}\Sigma_m \tag{3.5}$$

with explicit expressions for ε_E, ε_{T_l}, $\varepsilon_{k,E}$ and Σ_m given in (2.4), (2.14), (2.22) and (2.23), respectively.

In view of the relations (2.29) and (2.30), we further have

COROLLARY 1. *Under the same assumptions as Theorem 1, the following holds*

$$a_n(\theta) = \left(\frac{n}{2\theta}\right)^{\alpha-\frac{1}{2}} J_{\alpha-\frac{1}{2}}(n\theta) \sum_{k=0}^{m-1} \frac{\tilde{\alpha}_k(\theta)}{n^k} + \left(\frac{n}{2\theta}\right)^{\alpha-\frac{1}{2}} J_{\alpha+\frac{1}{2}}(n\theta) \sum_{k=0}^{m-1} \frac{\tilde{\beta}_k(\theta)}{n^k} + \tilde{\varepsilon}(\theta, m). \quad (3.6)$$

With α_k, β_k and $\varepsilon(\theta, m)$ replaced by $\tilde{\alpha}_k$, $\tilde{\beta}_k$ and $\tilde{\varepsilon}(\theta, m)$, respectively, the estimates (3.2), (3.3) and (3.4) remain valid.

Indeed, inserting (2.29) and (2.30) into (3.1) gives explicitly

$$\begin{aligned} \tilde{\alpha}_0(\theta) &= \sqrt{\pi}\alpha_0(\theta)/\Gamma(\alpha), \\ \tilde{\alpha}_k(\theta) &= (\sqrt{\pi}/\Gamma(\alpha))\left[\alpha_k(\theta) + (2\alpha-1)\beta_{k-1}(\theta)/\theta\right], \quad & k = 1, 2, 3, \cdots, \\ \tilde{\beta}_k(\theta) &= -\sqrt{\pi}\beta_k(\theta)/\Gamma(\alpha), & k = 0, 1, 2, \cdots, \end{aligned} \quad (3.7)$$

and

$$\tilde{\varepsilon}(\theta, m) = \varepsilon(\theta, m) + \frac{(2\alpha-1)\sqrt{\pi}}{\Gamma(\alpha)} \left(\frac{n}{2\theta}\right)^{\alpha-\frac{1}{2}} \frac{\beta_{m-1}(\theta)}{\theta} \frac{1}{n^m} J_{\alpha-\frac{1}{2}}(n\theta). \quad (3.8)$$

If in Corollary 1 we use (2.31), instead of (2.30), then $J_{\alpha+\frac{1}{2}}$ in (3.6) should be replaced by $J_{\alpha-\frac{3}{2}}$, and $\tilde{\alpha}_k, \tilde{\beta}_k$ can be replaced by α_k, β_k.

The remaining part of the present section is devoted to the proof of Theorem 1, and we shall proceed step by step.

3.2. Boundedness of the coefficients. The approach we take is stimulated by the work of Olde Daalhuis and Temme [9], where they have introduced several classes of rational functions in order to obtain rigorous error bounds for Airy-type uniform asymptotic expansions. For our purpose, we introduce two classes of rational functions associated with the iterative procedure (2.19) and (2.20). In fact, using Cauchy's integral formula and the fact that $h_0(s, \theta)$ is s-analytic in the region

$$D := \left\{ s \,\middle|\, \operatorname{Re} s \geq -\frac{\eta}{\theta}, \quad |s \pm i| < \frac{2\pi}{\theta} \right\}, \quad (3.9)$$

we have from (2.12)

$$\alpha_0(\theta) = \frac{1}{2\pi i} \int_{\mathcal{C}} A_0(s, \theta) h_0(s, \theta) ds \, , \quad \beta_0(\theta) = \frac{1}{2\pi i} \int_{\mathcal{C}} B_0(s, \theta) h_0(s, \theta) ds, \quad (3.10)$$

where $\mathcal{C}$ is a contour in D that encloses $s = \pm i$ in the positive sense (see Figure 2),

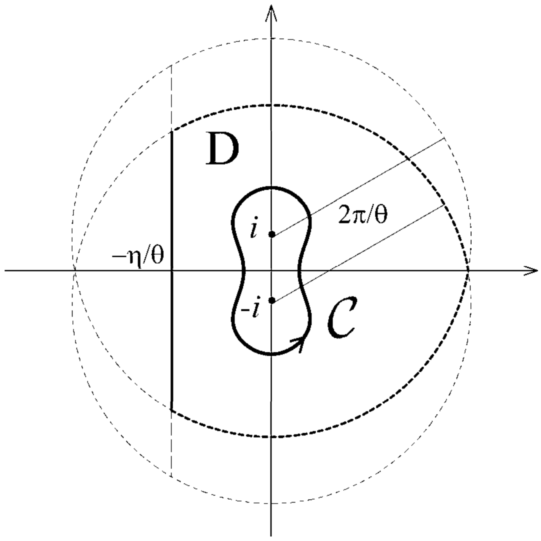

Figure 2. The domain D and the contour $\mathcal{C}$

$$A_0(s,\theta) = \frac{s}{s^2+1} \qquad \text{and} \qquad B_0(s,\theta) = \frac{1}{s^2+1}. \tag{3.11}$$

Define inductively

$$A_k(s,\theta) = \frac{1}{\theta}(1+s^2)^{-1}\left\{(s^2+1)\frac{d}{ds} + 2\alpha s\right\} A_{k-1}(s,\theta) \tag{3.12}$$

and

$$B_k(s,\theta) = \frac{1}{\theta}(1+s^2)^{-1}\left\{(s^2+1)\frac{d}{ds} + 2\alpha s\right\} B_{k-1}(s,\theta) \tag{3.13}$$

for $k = 1, 2, 3, \cdots$. The differentiation operator in (3.12) and (3.13) can of course be written as

$$\frac{1}{\theta}(s^2+1)^{-\alpha}\frac{d}{ds}\Big((s^2+1)^{\alpha} A_{k-1}(s,\theta)\Big).$$

In terms of these rational functions, we obtain the following representations.

Lemma 1. *For $\theta \in [0,\pi]$ and $k = 0, 1, 2, \cdots$, we have*

$$\alpha_k(\theta) = (1-2\alpha)\frac{\beta_{k-1}(\theta)}{\theta} + \frac{1}{2\pi i}\int_{\mathcal{C}} A_k(s,\theta) h_0(s,\theta)\,ds \tag{3.14}$$

and

$$\beta_k(\theta) = \frac{1}{2\pi i}\int_{\mathcal{C}} B_k(s,\theta) h_0(s,\theta)\,ds, \tag{3.15}$$

where $\mathcal{C}$ is the same contour as given in (3.10) and, for the sake of convenience, we have set $\beta_{-1} = 0$.

PROOF. We shall demonstrate only the result in (3.14). The corresponding result in (3.15) can be established in a similar manner. The case $k = 0$ is part of (3.10). For $k \geq 1$, we have from (2.19) and (2.20)

$$\begin{aligned}
\alpha_k(\theta) &= \frac{1}{2\pi i}\int_C A_0(s,\theta)h_k(s,\theta)ds \\
&= \frac{1}{2\pi i}\int_C A_0(s,\theta)\left\{-\frac{1}{\theta}(s^2+1)^{\alpha}\frac{d}{ds}\left[g_{k-1}(s,\theta)(s^2+1)^{1-\alpha}\right]\right\}ds \\
&= \frac{1}{2\pi i}\int_C A_1(s,\theta)(s^2+1)g_{k-1}(s,\theta)ds \\
&= \frac{1}{2\pi i}\int_C A_1(s,\theta)\left\{h_{k-1}(s,\theta)-\alpha_{k-1}(\theta)-s\beta_{k-1}(\theta)\right\}ds \\
&= \frac{1}{2\pi i}\int_C A_1(s,\theta)h_{k-1}(s,\theta)ds+(1-2\alpha)\frac{\beta_{k-1}(\theta)}{\theta} \\
&\cdots\cdots \\
&= \frac{1}{2\pi i}\int_C A_k(s,\theta)h_0(s,\theta)ds+(1-2\alpha)\frac{\beta_{k-1}(\theta)}{\theta},
\end{aligned}$$

thus proving (3.14). Here repeated use has been made of the facts that

$$\frac{1}{2\pi i}\int_C A_k(s,\theta)ds = 0, \qquad k = 1, 2, \cdots$$

$$\frac{1}{2\pi i}\int_C \theta s A_1(s,\theta)ds = (2\alpha - 1),$$

and

$$\frac{1}{2\pi i}\int_C \theta s A_k(s,\theta)ds = 0, \qquad k = 2, 3, \cdots,$$

which follow from (3.11), (3.12), and (3.17) below. ∎

LEMMA 2. *For $\theta \in (0,\pi]$, there exists a constant $M_k > 0$, independent of θ, such that*

$$|A_k(s,\theta)| \leq M_k\theta \quad \text{and} \quad |B_k(s,\theta)| \leq M_k\theta^2 \tag{3.16}$$

for $|s| \leq M/\theta$, $|s - i| \geq L/\theta$ and $|s + i| \geq L/\theta$, where L and M are positive constants.

PROOF. By induction, one can use (3.11) and (3.12) to write

$$A_k(s,\theta) = \frac{1}{\theta^k}\frac{p_{k+1}(s)}{(1+s^2)^{k+1}} \tag{3.17}$$

for $k = 0, 1, 2, \cdots$, where $p_{k+1}(s)$ is a polynomial of degree $k + 1$, with coefficients independent of θ. It can also be shown from (3.11) and (3.13) that

$$B_k(s,\theta) = \frac{1}{\theta^k}\frac{q_k(s)}{(1+s^2)^{k+1}} \tag{3.18}$$

for $k = 0, 1, 2, \cdots$, where $q_k(s)$ is a polynomial of degree k, independent of θ. The two inequalities in (3.16) now follow from (3.17) and (3.18), respectively. ∎

To estimate $h_0(s,\theta)$ in (2.9), we first recall that $f(e^{-\theta s},\theta)$ is uniformly bounded for $\theta \in [0,\pi]$ and $\operatorname{Re} s \geq -\eta/\theta$. Hence there exists a constant M_f, independent of θ and s, such that

$$\left|f(e^{-\theta s},\theta)\right| \leq M_f \quad \text{for} \quad \operatorname{Re} s \geq -\eta/\theta. \tag{3.19}$$

Next, since $(e^z-1)/z$ has no zero and is bounded on the circle $|z| = b$ for $0 < b < 2\pi$, there exist positive constants m_b and M_b such that

$$m_b \leq \left|\frac{e^z-1}{z}\right| \leq M_b \quad \text{for} \quad |z| \leq b.$$

Hence, for $0 < b < 2\pi$, we have

$$m_b \leq \left|\frac{e^{-s\theta}-e^{i\theta}}{(-s-i)\theta}\right| \leq M_b \quad \text{for} \quad |s+i| \leq \frac{b}{\theta}$$

and

$$m_b \leq \left|\frac{e^{-s\theta}-e^{-i\theta}}{(-s+i)\theta}\right| \leq M_b \quad \text{for} \quad |s-i| \leq \frac{b}{\theta}\,. \tag{3.20}$$

By combining (3.19), (3.20) and (2.9), we obtain

Lemma 3. *For $\theta \in (0,\pi]$, there exists a constant $M_D > 0$, independent of s and θ, such that*

$$|h_0(s,\theta)| \leq M_D \quad \textit{for} \quad \operatorname{Re} \quad s \geq -\frac{\eta}{\theta},\ |s+i| \leq \frac{b}{\theta} \textit{ and } |s-i| \leq \frac{b}{\theta}. \tag{3.21}$$

Now, for $\theta \in [0,\pi-\delta]$, one may specify $b = 2\pi-\delta$ in (3.21). Without loss of generality, we may always assume that $\eta < \sqrt{\pi(3\pi-2\delta)}$. The contour $\mathcal{C}$ in (3.10), (3.14) and (3.15) may be deformed so that it consists of i) $|s+i| = b/\theta$, $\operatorname{Im} s \geq 0$ and $\operatorname{Re} s \geq -\eta/\theta$; ii) $|s-i| = b/\theta$, $\operatorname{Im} s \leq 0$ and $\operatorname{Re} s \geq -\eta/\theta$; and iii) the segment of $\operatorname{Re} s = -\eta/\theta$ joining i) and ii); see Figure 3. The constants M and L in Lemma 2 may be chosen to be $M = \max\{\eta, 3\pi-2\delta\}$ and $L = \min\{\eta,\delta\}$. A combination of Lemmas 1-3 then gives the boundedness of the coefficients $\alpha_k(\theta)$ and $\beta_k(\theta)/\theta$, i.e., (3.2) and (3.3), thus proving that part of Theorem 1.

3.3. Boundedness of the error term. To describe the behavior of $T_1(n\theta)$ and $T_2(n\theta)$, we make use of (2.29) and (2.30). From the behavior of $J_{\alpha-\frac{1}{2}}(n\theta)$ and $J_{\alpha+\frac{1}{2}}(n\theta)$ when $n\theta$ is small, we have

$$T_1(n\theta) \sim \frac{1}{\Gamma(2\alpha)}(n\theta)^{2\alpha-1}, \quad \text{as} \quad n\theta \to 0+,$$

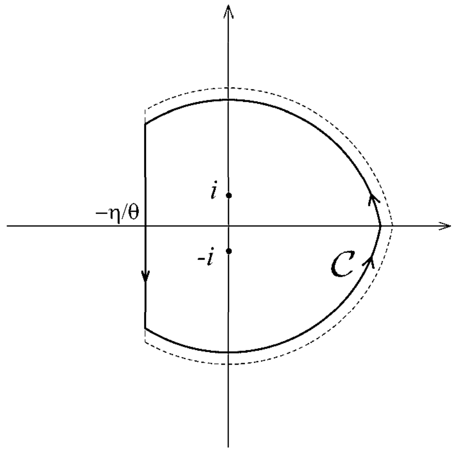

Figure 3. The deformed contour $\mathcal{C}$

and

$$T_2(n\theta) \sim \frac{1}{\Gamma(2\alpha-1)}(n\theta)^{2\alpha-2} - \frac{2\alpha}{\Gamma(2\alpha+2)}(n\theta)^{2\alpha-1}, \quad \text{as} \ \ n\theta \to 0+;$$

see, e.g., Wong [13, p. 231]. Hence

$$|T_1(n\theta)| + |T_2(n\theta)| \geq C \cdot (n\theta)^{2\alpha-1} \tag{3.22}$$

for $n\theta \in [0, \epsilon]$ and ϵ small, where C depends only on ϵ. The interval of validity for (3.22) can of course be extended to $n\theta \in [0, B]$ for a finite B, since $J_{\alpha-\frac{1}{2}}(\tau)$ and $J_{\alpha+\frac{1}{2}}(\tau)$ have no common zeros. The constant C may then depend on B.

In view of the behavior of the Bessel function (see, e.g., Wong [13, p. 206]), we again have from (2.29) and (2.30)

$$T_1(n\theta) \sim \frac{1}{\Gamma(\alpha)}\left(\frac{n\theta}{2}\right)^{\alpha-1} \cos\left(n\theta - \frac{1}{2}\alpha\pi\right), \quad \text{as} \ \ n\theta \to +\infty$$

and

$$T_2(n\theta) \sim \frac{1}{\Gamma(\alpha)}\left(\frac{n\theta}{2}\right)^{\alpha-1}\left[\sin\left(n\theta - \frac{1}{2}\alpha\pi\right) + \frac{2\alpha-1}{n\theta}\cos\left(n\theta - \frac{1}{2}\alpha\pi\right)\right]$$

as $n\theta \to +\infty$. Hence

$$|T_1(n\theta)| + |T_2(n\theta)| \geq C(n\theta)^{\alpha-1} \tag{3.23}$$

for $n\theta \in [B, \infty)$, where B is a large but fixed number.

To estimate the error terms, we note from (2.27) that

$$|\varepsilon_{T_1}| \leq \frac{C(\eta)}{n}\theta^{2\alpha-1}e^{-\eta m} = C(\eta)\left\{n^{m-2\alpha}e^{-\eta m}\right\}\frac{(n\theta)^{2\alpha-1}}{n^m} \leq C\frac{(n\theta)^{2\alpha-1}}{n^m}$$

for $n\theta \in [0, B]$, and from (2.28) that

$$\theta|\varepsilon_{T_2}| \leq C\frac{(n\theta)^{2\alpha-1}}{n^m}$$

also for $n\theta \in [0, B]$.

When $n\theta \in [B, \infty)$, and hence for $\theta \in [B/n, \pi]$, it follows from (2.27) that

$$|\varepsilon_{T_1}| \leq C\left\{\frac{\theta^\alpha}{n^\alpha}e^{-\eta n}\right\}(n\theta)^{\alpha-1} \leq C\left\{n^{m+|\alpha|-\alpha}e^{-\eta n}\right\}\frac{(n\theta)^{\alpha-1}}{n^m} \leq C\frac{(n\theta)^{\alpha-1}}{n^m}.$$

Similarly, from (2.28) we have

$$\theta|\varepsilon_{T_2}| \leq C\frac{(n\theta)^{\alpha-1}}{n^m}.$$

Summarizing the last four inequalities, we obtain, in view of (3.22) and (3.23),

$$\theta^{l-1}|\varepsilon_{T_l}| \leq C_m\frac{1}{n^m}\left\{|T_1(n\theta)| + |T_2(n\theta)|\right\}, \quad l = 1, 2,$$

where C_m is a constant independent of n and θ. Accordingly,

$$\left|\theta^{1-2\alpha}\sum_{k=0}^{m-1}\frac{\alpha_k(\theta)\varepsilon_{T_1} + \beta_k(\theta)\varepsilon_{T_2}}{n^k}\right| \leq C\frac{\theta^{1-2\alpha}}{n^m}\left\{|T_1(n\theta)| + |T_2(n\theta)|\right\} \tag{3.24}$$

for all n and θ, where use has been made of the estimates (3.2) and (3.3).

An estimate for ε_E can be obtained by comparing (2.5) with (3.22) and (3.23). Since

$$e^{-\eta n} = \left\{n^{m-2\alpha+1}e^{-\eta n}\right\}\frac{\theta^{1-2\alpha}(n\theta)^{2\alpha-1}}{n^m} \leq C\frac{\theta^{1-2\alpha}(n\theta)^{2\alpha-1}}{n^m}$$

for $n\theta \in [0, B]$, and

$$e^{-\eta n} \leq C\left\{n^{m-\alpha+|\alpha|+1}e^{-\eta n}\right\}\frac{\theta^{1-2\alpha}(n\theta)^{\alpha-1}}{n^m} \leq C\frac{\theta^{1-2\alpha}(n\theta)^{\alpha-1}}{n^m}$$

for $n\theta \in [B, \infty)$ (and hence $\theta \in [B/n, \pi]$), it follows that

$$|\varepsilon_E| \leq M_m\frac{\theta^{1-2\alpha}}{n^m}\left[|T_1(n\theta)| + |T_2(n\theta)|\right]. \tag{3.25}$$

To investigate $\varepsilon_{k,E}$ and Σ_m, we first analyze $h_k(s, \theta)$ and $g_k(s, \theta)$. Let us introduce another class of rational functions associated with (2.19) and (2.20). By Cauchy's theorem,

$$h_0(s, \theta) = \frac{1}{2\pi i}\int_{\mathcal{C}_u}\frac{h_0(u, \theta)}{u - s}\,du,$$

where the integration path $\mathcal{C}_u$ is a contour that lies in the domain D of u-analyticity (see Figure 2), and encloses $u = s$ and $u = \pm i$ in the counterclockwise direction. Set

$$Q_0(u, s, \theta) = \frac{1}{u - s}. \tag{3.26}$$

Then

$$h_0(s,\theta) = \frac{1}{2\pi i}\int_{\mathcal{C}_u} Q_0(u,s,\theta)h_0(u,\theta)\,du. \tag{3.27}$$

We further define

$$Q_k(u,s,\theta) = \frac{1}{\theta}\left[\frac{d}{du} + 2\alpha\frac{u}{u^2+1}\right] Q_{k-1}(u,s,\theta), \quad k=1,2,3,\cdots; \tag{3.28}$$

see the comment following (3.13).

In view of (3.26) and (3.28), it can be shown by induction that

$$Q_k(u,s,\theta) = \frac{1}{\theta^k}\sum_{l=0}^{k}\frac{P_l(u)}{(u-s)^{k-l+1}(1+u^2)^l}, \tag{3.29}$$

where $P_l(u)$ is a polynomial of degree l whose coefficients are independent of u, s and θ. The last equation suggests that

$$\frac{1}{2\pi i}\int_{\mathcal{C}_u} Q_k(u,s,\theta)du = 0, \qquad k=1,2,3,\cdots,$$

$$\frac{1}{2\pi i}\int_{\mathcal{C}_u} \theta u Q_1(u,s,\theta)du = 2\alpha - 1,$$

and

$$\frac{1}{2\pi i}\int_{\mathcal{C}_u} uQ_k(u,s,\theta)du = 0, \qquad k=2,3,4,\cdots.$$

Similar to the derivation of (3.14) and (3.15), we have

Lemma 4. *For $\theta \in [0,\pi]$ and $k=0,1,2,\cdots$,*

$$h_k(s,\theta) = (1-2\alpha)\frac{\beta_{k-1}(\theta)}{\theta} + \frac{1}{2\pi i}\int_{\mathcal{C}_u} Q_k(u,s,\theta)h_0(u,\theta)du, \tag{3.30}$$

where, for convenience, we have set $\beta_{-1}(\theta) = 0$.

From (3.29), one can also see that the following estimates hold.

Lemma 5. *For $\theta \in (0,\pi]$, $|u| \le M/\theta$, $|s| \le M/\theta$, $|u-s| \ge L/\theta$, $|u-i| \ge L/\theta$ and $|u+i| \ge L/\theta$, there exist constants $M_k, k=0,1,2,\cdots$, such that*

$$|Q_k(u,s,\theta)| \le M_k\theta. \tag{3.31}$$

Choose a s-contour Γ_s similar to $\mathcal{C}$, described in the paragraph following Lemma 3. Γ_s consists of i) $|s+i| = b/\theta$, $\operatorname{Im} s \ge 0$ and $\operatorname{Re} s \ge -(\eta-\epsilon)/\theta$; ii) $|s-i| = b/\theta$, $\operatorname{Im} s \le 0$ and $\operatorname{Re} s \ge -(\eta-\epsilon)/\theta$; and iii) the segment of $\operatorname{Re} s = -(\eta-\epsilon)/\theta$ joining i) and ii), where ϵ is a positive number which is sufficiently small so that Γ_s encloses $\pm i$; see Figure 4.

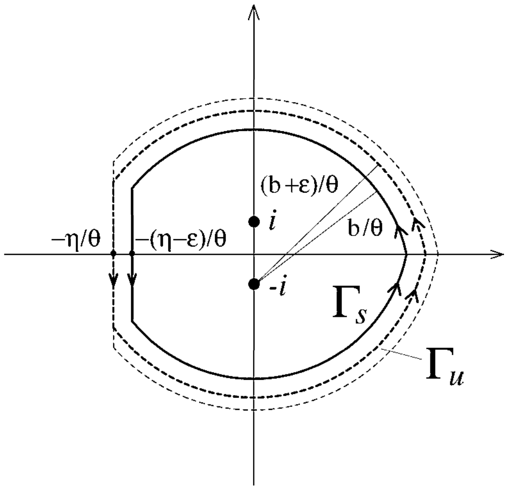

Figure 4. The contour Γ_s and Γ_u

Similarly, we define Γ_u, consisting of i) $|s+i| = (b+\epsilon)/\theta$, $\operatorname{Im} s \geq 0$ and $\operatorname{Re} s \geq -\eta/\theta$; ii) $|s-i| = (b+\epsilon)/\theta$, $\operatorname{Im} s \leq 0$ and $\operatorname{Re} s \geq -\eta/\theta$; and iii) the segment of $\operatorname{Re} s = -\eta/\theta$ joining i) and ii). Denote by D_S the domain bounded by Γ_s. If $s \in D_S$ and $u \in \Gamma_u$, then Lemma 3 holds with s replaced by u, and Lemmas 4 and 5 hold since Γ_u encloses $u = s$ and $u = \pm i$, and lies in D, and since it follows from the previous description that $|u - s| \geq \epsilon/\theta$.

We notice that in the previous derivation leading to (2.21) and the estimation leading to (3.2), (3.3), (3.24) and (3.25), we only require that η be a fixed positive number. Hence, in these cases we can replace η by a smaller number, say, $\eta' = \eta - \epsilon$, and the validity of these previous results will remain. For convenience, let us continue to denote the smaller η' by η. With this understanding, one obtains the following result by combining Lemmas 3-5 and using the fact that $\int_{\Gamma_u} |du| = O(1/\theta)$.

LEMMA 6. *For $\theta \in (0, \pi]$ and $k = 0, 1, 2, \cdots$, we have*

$$|h_k(s,\theta)| \leq M_k, \qquad s \in D_S, \tag{3.32}$$

where D_S is the domain bounded by i) $|s+i| = b/\theta$, $\operatorname{Re} s \geq -\eta/\theta$ and $\operatorname{Im} s \geq 0$, ii) $|s-i| = b/\theta$, $\operatorname{Re} s \geq -\eta/\theta$ and $\operatorname{Im} s \leq 0$, and iii) $\operatorname{Re} s = -\eta/\theta$, $|\operatorname{Im} s| \leq \sqrt{b^2 - \eta^2}/\theta$.

We are now ready to consider the term $\varepsilon_{k,E}$ given in (2.22). By (2.19),

$$g_{k-1}(s,\theta)(s^2+1)^{1-\alpha}e^{ns\theta} = \left[h_{k-1}(s,\theta) - \alpha_{k-1}(\theta) - s\frac{\beta_{k-1}(\theta)}{\theta}\theta\right](s^2+1)^{-\alpha}e^{ns\theta}.$$

Since $\eta^2/\theta^2 < \eta^2/\theta^2 + 1 < (\eta^2+\pi^2)/\theta^2$, $(s^2+1)^{-\alpha}\Big|_{s=e^{\pm i\pi}\eta/\theta}$ is bounded by $C(\eta)\theta^{2\alpha}$. In view of the boundedness of h_{k-1}, $\alpha_{k-1}(\theta)$ and $\beta_{k-1}(\theta)/\theta$, it follows that

$$|\varepsilon_{k,E}| \leq C(\eta, M_{k-1})e^{-\eta n}. \tag{3.33}$$

Using the inequalities preceding (3.25), one can show that the estimate for ε_E in (3.25) also holds for $\varepsilon_{k,E}, k = 1, 2, \cdots$. Hence, we have

$$\left|\sum_{k=1}^{m} \frac{\varepsilon_{k,E}}{n^k}\right| \le M_m \frac{\theta^{1-2\alpha}}{n^m} \left[|T_1(n\theta)| + |T_2(n\theta)|\right]. \tag{3.34}$$

The only remaining task in the present section is to estimate Σ_m given in (2.23). For $n\theta \in [0, B]$, we deform the integration path Γ so that it starts from $e^{-i\pi}\eta/\theta$ and ends at $e^{i\pi}\eta/\theta$, and that there are positive constants L and M such that $|s \pm i| \ge L/\theta$ and $|s| \le M/\theta$ along Γ; for an example of such paths, see the paragraph following Lemma 3. Now make the change of variable $n\theta s = t$, and denote the image of the s-curve Γ by $\tilde{\Gamma}_t$. It is readily seen that $\tilde{\Gamma}_t$ is a curve which starts at $e^{-i\pi}\eta n$ and ends at $e^{i\pi}\eta n$; along $\tilde{\Gamma}_t$, we have $|t \pm in\theta| \ge nL$, $|t| \le nM$ and

$$\Sigma_m = \frac{\theta^{1-2\alpha}}{2\pi i}(n\theta)^{2\alpha-1} \int_{\tilde{\Gamma}_t} h_m(s,\theta)\left(t^2 + (n\theta)^2\right)^{-\alpha} e^t dt. \tag{3.35}$$

We further deform $\tilde{\Gamma}_t$ so that it traverses from $e^{-i\pi}\eta n$ to $e^{-i\pi}(2B)$ along the lower edge of the negative real line, moves to $e^{i\pi}(2B)$ on the circle $|t| = 2B$ in the counterclockwise direction, and then along the upper edge of the negative real line to $e^{i\pi}\eta n$. The deformed curve will still be denoted by $\tilde{\Gamma}_t$. Along this new curve, $\left|(t^2 + (n\theta)^2)^{-\alpha}\right| \le C(B)t^{-2\alpha}$ and $|h_m(s,\theta)| \le M_m$; cf. (3.32). Thus,

$$\begin{aligned} |\Sigma_m| &\le \theta^{1-2\alpha} C(M_m, B)(n\theta)^{2\alpha-1} \int_{\tilde{\Gamma}_t} \left|t^{-2\alpha} e^t\right| |dt| \\ &\le C(M_m, B)\theta^{1-2\alpha}(n\theta)^{2\alpha-1} \end{aligned} \tag{3.36}$$

for $n\theta \in [0, B]$. In view of (3.22), we obtain

$$|\Sigma_m(\theta)| \le M_m \theta^{1-2\alpha}\left[|T_1(n\theta)| + |T_2(n\theta)|\right] \tag{3.37}$$

for $n\theta \in [0, B]$, $m = 1, 2, 3, \cdots$.

Finally we consider the case when $n\theta \to +\infty$. First, we introduce a curve Γ_c depending on $n\theta$, which starts at $e^{-i\pi}\eta/\theta$, moves to $e^{-i\pi}/n\theta$ along the lower edge of the negative real axis, encircles the origin along the circle $|s| = 1/n\theta$ in the positive sense, and then proceeds from $e^{i\pi}/n\theta$ to $e^{i\pi}\eta/\theta$ along the upper edge of the negative real line. We now deform the path of integration in (2.23), and split it into three parts: $\Gamma_i = \Gamma_c + i$, $\Gamma_{-i} = \Gamma_c - i$ and Γ_r, where Γ_r consists of three segments on $\mathrm{Re}\, s = -\eta/\theta$ connecting i) $e^{i\pi}\eta/\theta - i$ and $e^{-i\pi}\eta/\theta + i$, ii) $e^{i\pi}\eta/\theta + i$ and $e^{i\pi}\eta/\theta$, and iii) $e^{-i\pi}\eta/\theta - i$ and $e^{-i\pi}\eta/\theta$; see Figure 5.

We know from Lemma 6 that $h_m(s,\theta)$ is bounded on $\Gamma = \Gamma_i \bigcup \Gamma_{-i} \bigcup \Gamma_r$, and that the bound is uniform in $\theta \in [0, \pi - \delta]$. Consider

$$I_i \equiv \frac{1}{2\pi i}\int_{\Gamma_i} h_m(s,\theta)(s^2+1)^{-\alpha} e^{n\theta s} ds = \frac{e^{in\theta}}{2\pi i}\int_{\Gamma_c} \left\{h_m(s+i,\theta)(s+2i)^{-\alpha}\right\} s^{-\alpha} e^{n\theta s} ds.$$

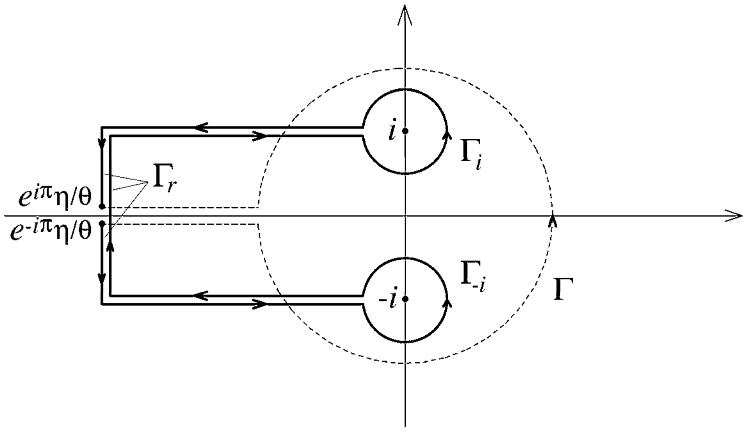

Figure 5. Contour $\Gamma = \Gamma_i \bigcup \Gamma_{-i} \bigcup \Gamma_r$

To obtain an estimate for I_i, our argument is similar to that used by Olver [10, pp. 71-72] to prove Watson's lemma. For the purpose of completeness, we give a brief account of this argument. In the last integral we put $v(s) \equiv h_m(s+i,\theta)(s+2i)^{-\alpha}$ and make the change of variable $t = n\theta s$. Since $v(s)$ is uniformly bounded on $\{\Gamma_c : \text{Re}\, s \geq -3\}$, we have

$$\begin{aligned}\left|\frac{e^{in\theta}}{2\pi i}\int_{\{\Gamma_c:\text{Re}\, s\geq -3\}} v(s)s^{-\alpha}e^{n\theta s}ds\right| &= (n\theta)^{\alpha-1}\left|\frac{e^{in\theta}}{2\pi i}\int_{-3n\theta}^{(0+)} v(s(t))t^{-\alpha}e^{t}dt\right| \\ &\leq \left\{C\int_{-\infty}^{(0+)} |t|^{-\alpha}e^{\text{Re}\, t}|dt|\right\}(n\theta)^{\alpha-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|\frac{e^{in\theta}}{2\pi i}\int_{\{\Gamma_c:\text{Re}\, s\leq -3\}} w(s)e^{n\theta s}ds\right| &\leq \frac{M_m C(\alpha)}{\pi}\int_3^{\eta/\theta} t^{-2\alpha}e^{-n\theta t}dt \\ &\leq Ce^{-3n\theta},\end{aligned}$$

which is in turn bounded by $(n\theta)^{\alpha-1}$ for $n\theta \geq B$. The second last inequality follows from the fact that $w(s) \equiv h_m(s+i,\theta)(s+2i)^{-\alpha}s^{-\alpha}$ is uniformly bounded by $C|s|^{-2\alpha}$ on that part of Γ_c. Hence

$$|I_i| \leq C(n\theta)^{\alpha-1}. \tag{3.38}$$

Similarly, we have

$$|I_{-i}| \leq C(n\theta)^{\alpha-1}. \tag{3.39}$$

The estimate of the integral I_r over Γ_r can be obtained by taking the absolute value of the integrand. Indeed, we have

$$|I_r| \leq Ce^{-\eta n}\theta^{-2\alpha} \leq C(n\theta)^{\alpha-1}. \tag{3.40}$$

To obtain the last inequality, we have used the fact that $\theta \in [B/n, \pi]$ for $n\theta \in [B, \infty)$. A combination of (3.38), (3.39), (3.40) and the fact that $\Sigma_m = \theta^{1-2\alpha}(I_i + I_{-i} + I_r)$ gives

$$|\Sigma_m(\theta)| \leq C\theta^{1-2\alpha}(n\theta)^{\alpha-1} \leq C\theta^{1-2\alpha}\left\{|T_1(n\theta)| + |T_2(n\theta)|\right\} \tag{3.41}$$

for $n\theta \in [B, +\infty)$, B being sufficiently large, $m = 1, 2, 3, \cdots$; see (3.23). The results in (3.37) and (3.41) imply that there exists a constant C such that

$$|\Sigma_m(\theta)| \leq C\theta^{1-2\alpha}\left\{|T_1(n\theta)| + |T_2(n\theta)|\right\} \tag{3.42}$$

for all n and θ. The desired result (3.4) now follows from (3.24), (3.25), (3.34), (3.42) and (3.5).

4 Many coalescing algebraic singularities

4.1. Formal derivation. In this subsection, we give a brief description of the general case when there are two or more branch points on the circle of convergence that coalesce with each other when an auxiliary parameter approaches some critical value. Typically, we assume that the generating function and its Maclaurin expansion take the form

$$F(z,\theta) = \left\{\prod_{k=1}^{q}(z_k(\theta) - z)^{-\alpha_k}\right\} f(z,\theta) = \sum_{n=0}^{\infty} a_n(\theta) z^n, \tag{4.1}$$

where $|z_k(\theta)| = 1$, α_k are constants (for the sake of convenience, we assume that all these quantities are real), and $f(z,\theta)$ is an analytic function of z in $|z| \leq e^{\eta}$, with $\eta > 0$ being a constant independent of θ. Also, we assume that $f(z,\theta)$ is subject to some smoothness and uniform boundedness conditions, and that

$$z_k(\theta) \longrightarrow 1 \qquad \text{as} \qquad \theta \to 0. \tag{4.2}$$

The problem is to derive an asymptotic expansion for $a_n(\theta)$ as $n \to \infty$, which holds uniformly for θ in some interval containing $\theta = 0$.

If we further require that each $z_k(\theta)$ be continuously differentiable with respect to θ in a neighborhood of $\theta = 0$, then by our assumption we can write $z_k(\theta) \equiv e^{i\theta s_k(\theta)}$, where each $s_k(\theta)$ is real and $s_k(0) = -iz_k'(0)$.

As in (2.2), Cauchy's formula gives

$$a_n(\theta) = \frac{1}{2\pi i}\int_C f(z,\theta)\left\{\prod_{k=1}^{q}(z_k(\theta) - z)^{-\alpha_k}\right\} z^{-n-1}dz, \tag{4.3}$$

where C is a closed curve that encloses the origin but not the branch points $z = z_k(\theta), k = 1, \cdots, q$. For later use, we may choose C such that it consists of two closed contours: i) C_E, the circle $|z| = e^{\eta}$, oriented counterclockwise and ii) C_I, a clockwisely oriented contour that starts and ends at $z = e^{\eta}$, and encloses all the singularities $z_k(\theta), k = 1, \cdots, q$, but not $z = 0$; see Figure 6. (C_I need not be a circle.)

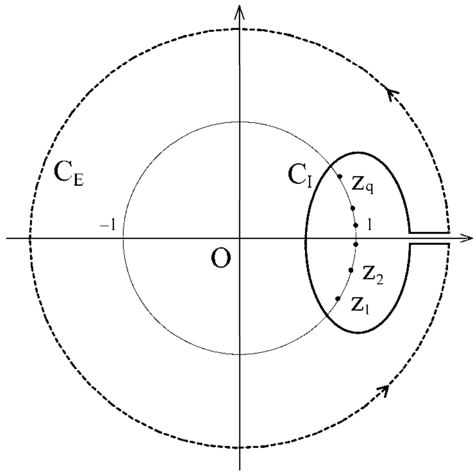

Figure 6. The contour in (4.3)

Denote by $A_n(\theta)$ and $\varepsilon_E(\theta)$ respectively the integral in (4.3) over C_E and C_I. Using the transformation (2.7), that is

$$z = e^{-\theta s}, \tag{4.4}$$

the integral $A_n(\theta)$ can be expressed as

$$A_n(\theta) = \frac{\theta^{1-\alpha}}{2\pi i} \int_\Gamma h_0(s,\theta) \left\{ \prod_{k=1}^{q} (s + is_k(\theta))^{-\alpha_k} \right\} e^{n\theta s} ds, \tag{4.5}$$

where $\alpha = \sum_{k=1}^{q} \alpha_k$,

$$h_0(s,\theta) = \left\{ \prod_{k=1}^{q} \left(\frac{e^{i\theta s_k(\theta)} - e^{-\theta s}}{i\theta s_k(\theta) + \theta s} \right)^{-\alpha_k} \right\} f(e^{-\theta s}, \theta), \tag{4.6}$$

and Γ is the image of C_I under transformation (4.4), which starts from $s = e^{-i\pi}\eta/\theta$, encircles all points $s = -is_k(\theta)$ in the counterclockwise direction, and ends at $s = e^{i\pi}\eta/\theta$.

To derive the formal expansion of $a_n(\theta)$, we introduce the functions

$$T_l(x) \equiv \frac{1}{2\pi i} \int_{\Gamma_0} s^{l-1} \left\{ \prod_{k=1}^{q} (s + is_k(\theta))^{-\alpha_k} \right\} e^{xs} ds \tag{4.7}$$

$l = 1, 2, \cdots, q$, where Γ_0 is a loop of Hankel type, which starts from $-\infty$, encircles all points $s = -is_k(\theta)$ in the positive direction, and ends at $-\infty$. Note that $T_l(x)$ is the $(l-1)th$ derivative of $T_1(x)$; see (4.49). Now write

$$h_0(s,\theta) = \beta_{0,1}(\theta) + \beta_{0,2}(\theta)s + \cdots + \beta_{0,q}(\theta)s^{q-1} + \left\{ \prod_{k=1}^{q} (s + is_k(\theta)) \right\} g_0(s,\theta). \tag{4.8}$$

From (4.3) and (4.5), it follows by integration by parts

$$a_n(\theta) = \theta^{1-\alpha} \sum_{l=1}^{q} \beta_{0,l}(\theta) T_l(n\theta) + \left\{ \varepsilon_E + \frac{1}{n}\varepsilon_{1,E} - \theta^{1-\alpha} \sum_{l=1}^{q} \beta_{0,l}(\theta) \varepsilon_{T_l} + \frac{1}{n}\Sigma_1 \right\}, \qquad (4.9)$$

where ε_E is given in the paragraph containing (4.3),

$$\varepsilon_{T_l} = \frac{1}{2\pi i} \int_{\Gamma_0 \setminus \Gamma} s^{l-1} \left\{ \prod_{k=1}^{q} (s + i s_k(\theta))^{-\alpha_k} \right\} e^{xs} ds, \qquad (4.10)$$

$$\varepsilon_{1,E} = \frac{\theta^{-\alpha}}{2\pi i} g_0(s,\theta) \left\{ \prod_{k=1}^{q} (s + i s_k(\theta))^{1-\alpha} \right\} e^{n\theta s} \Bigg|_{s=e^{-i\pi}\eta/\theta}^{s=e^{i\pi}\eta/\theta}, \qquad (4.11)$$

$$\Sigma_1 = \frac{\theta^{1-\alpha}}{2\pi i} \int_{\Gamma} h_1(s,\theta) \left\{ \prod_{k=1}^{q} (s + i s_k(\theta))^{-\alpha_k} \right\} e^{n\theta s} ds \qquad (4.12)$$

and

$$h_1(s,\theta) = \frac{1}{\theta} \prod_{k=1}^{q} (s + i s_k(\theta)) \left\{ -\frac{d}{ds} + \sum_{l=1}^{q} \frac{\alpha_l - 1}{s + i s_l(\theta)} \right\} g_0(s,\theta). \qquad (4.13)$$

This procedure can be repeated, and we define inductively

$$h_j(s,\theta) = \sum_{l=1}^{q} \beta_{j,l}(\theta) s^{l-1} + \left\{ \prod_{k=1}^{q} (s + i s_k(\theta)) \right\} g_j(s,\theta) \qquad (4.14)$$

and

$$h_{j+1}(s,\theta) = \frac{1}{\theta} \prod_{k=1}^{q} (s + i s_k(\theta)) \left\{ -\frac{d}{ds} + \sum_{l=1}^{q} \frac{\alpha_l - 1}{s + i s_l(\theta)} \right\} g_j(s,\theta), \qquad (4.15)$$

$j = 0, 1, 2, \cdots$. The following expansion is then derived, with its coefficients and remainder expressed in terms of $h_j(s,\theta)$ and $g_j(s,\theta)$:

$$a_n(\theta) = \theta^{1-\alpha} \sum_{l=1}^{q} T_l(n\theta) \sum_{k=0}^{m-1} \frac{\beta_{k,l}(\theta)}{n^k} + \varepsilon(\theta, m) \qquad (4.16)$$

for $m = 1, 2, 3, \cdots$, where

$$\varepsilon(\theta, m) = \varepsilon_E + \sum_{l=1}^{m} \frac{\varepsilon_{l,E}}{n^l} - \theta^{1-\alpha} \sum_{l=1}^{q} \varepsilon_{T_l} \sum_{k=0}^{m-1} \frac{\beta_{k,l}(\theta)}{n^k} + \frac{1}{n^m}\Sigma_m \qquad (4.17)$$

and ε_{T_l} is given in (4.10). The other remainders ε_E, $\varepsilon_{l,E}$ and Σ_m are given explicitly by

$$\varepsilon_E = \frac{1}{2\pi i} \int_{C_E} f(z,\theta) \left\{ \prod_{k=1}^{q} (z_k(\theta) - z)^{-\alpha_k} \right\} z^{-n-1} dz, \qquad (4.18)$$

$$\varepsilon_{l,E} = \frac{\theta^{-\alpha}}{2\pi i} g_{l-1}(s,\theta) \left\{ \prod_{k=1}^{q} (s + is_k(\theta))^{1-\alpha} \right\} e^{n\theta s} \Bigg|_{s=e^{-i\pi}\eta/\theta}^{s=e^{i\pi}\eta/\theta} \tag{4.19}$$

and

$$\Sigma_m = \frac{\theta^{1-\alpha}}{2\pi i} \int_\Gamma h_m(s,\theta) \left\{ \prod_{k=1}^{q} (s + is_k(\theta))^{-\alpha_k} \right\} e^{n\theta s} ds. \tag{4.20}$$

The above derivation is in a sense motivated by that of Bleistein [1]. But in [1], the relevant integrals involve saddle points of the phase function, whereas in the present case, there is no saddle point at all. The key part of the present paper lies in the error estimate, which makes the expansions in (3.1) and (4.16) uniformly asymptotic. Although the analysis of the estimation is quite complicated, the derivation itself is relatively straightforward.

To illustrate that our formal derivation can be made rigorous, let us consider the special case

$$s_k(\theta) \equiv s_k, \quad k = 1, 2, \ldots, q, \qquad\qquad s_k \neq s_l \quad \text{for} \quad k \neq l, \tag{4.21}$$

or, equivalently, $z_k(\theta) = e^{is_k\theta}$ in (4.1). Here, we do not intend to give the details of the analysis, but will provide the key steps and facts.

4.2. Classes of rational functions. Putting $s = -is_k$ into (4.14) yields

$$\sum_{l=1}^{q} (-is_k)^{l-1} \beta_{j,l}(\theta) = h_j(-is_k, \theta), \quad k = 1, 2, \cdots, q. \tag{4.22}$$

Let $m_{k,l} = (-is_k)^{l-1}$ for $k, l = 1, 2, \cdots, q$, and

$$\mathbf{M} = (m_{k,l})_{q\times q}, \quad \boldsymbol{\beta}_j = (\beta_{j,l})_{q\times 1}, \quad \text{and} \quad \mathbf{h}_j = (h_j(-is_k,\theta))_{q\times 1}. \tag{4.23}$$

Then, (4.22) can be written in the matrix form

$$\mathbf{M}\boldsymbol{\beta}_j = \mathbf{h}_j$$

or, equivalently,

$$\boldsymbol{\beta}_j = \mathbf{M}^{-1}\mathbf{h}_j \tag{4.24}$$

since $\mathbf{M}$ is nonsingular when $s_k \neq s_{k'}$ for $k \neq k'$. Writing

$$\mathbf{M}^{-1} = (\tilde{m}_{l,k})_{q\times q}, \tag{4.25}$$

one has

$$\beta_{j,l}(\theta) = \sum_{k=1}^{q} \tilde{m}_{l,k}\, h_j(-is_k, \theta) = \frac{1}{2\pi i} \int_C h_j(s,\theta) \sum_{k=1}^{q} \frac{\tilde{m}_{l,k}}{s + is_k} ds, \tag{4.26}$$

where $\mathcal{C}$ is a closed contour that encloses all the poles $s = -is_k, k = 1, \cdots, q$. Define

$$A_{0,l}(s,\theta) = \sum_{k=1}^{q} \frac{\tilde{m}_{l,k}}{s + is_k}, \quad l = 1, 2, \cdots, q. \tag{4.27}$$

Then (4.26) implies that

$$\beta_{0,l}(\theta) = \frac{1}{2\pi i} \int_{\mathcal{C}} A_{0,l}(s,\theta)\, h_0(s,\theta) ds. \tag{4.28}$$

Inductively, we define

$$A_{j,l}(s,\theta) = \frac{1}{\theta} \prod_{k=1}^{q} (s + is_k)^{-\alpha_k} \frac{d}{ds} \left[A_{j-1,l}(s,\theta) \prod_{k=1}^{q} (s + is_k)^{\alpha_k} \right] \tag{4.29}$$

for $l = 1, 2, \cdots, q$ and $j = 1, 2, \cdots$. The sequences $\{A_{j,l}\}_{j=0}^{\infty}$, $l = 1, 2, \cdots, q$, form q classes of rational functions. In fact, in view of (4.23), (4.25) and (4.27), we have

$$A_{0,l}(s,\theta) = \frac{1}{\det \mathbf{M}} \begin{vmatrix} 1 & -is_1 & \cdots & 1/(s+is_1) & \cdots & (-is_1)^{q-1} \\ 1 & -is_2 & \cdots & 1/(s+is_2) & \cdots & (-is_2)^{q-1} \\ & \cdots & & & & \cdots \\ 1 & -is_q & \cdots & 1/(s+is_q) & \cdots & (-is_q)^{q-1} \end{vmatrix}. \tag{4.30}$$

The determinant on the right-hand side of this equation is obtained by replacing the l-th column of the matrix $\mathbf{M}$ in (4.23) by the vector $(1/(s + is_k))_{q\times 1}$. Inserting

$$\frac{1}{s + is_k} = \sum_{j'=1}^{q} \frac{(-is_k)^{j'-1}}{s^{j'}} + \frac{1}{s^q} \frac{(-is_k)^q}{s + is_k}$$

into (4.30), we obtain

$$A_{0,l}(s,\theta) = \frac{1}{s^l} + \frac{1}{s^q} \sum_{k=1}^{q} \frac{\tilde{m}_{l,k}\,(-is_k)^q}{s + is_k},$$

and, on account of (4.27),

$$A_{0,l}(s,\theta) = \frac{P_{q-l}(s)}{\prod_{k=1}^{q}(s + is_k)}, \qquad l = 1, 2, \cdots, q, \tag{4.31}$$

where P_{q-l} denotes a polynomial of degree at most $q - l$.

From the iterative relation (4.29), it can be deduced that

$$A_{j,l}(s,\theta) = \frac{1}{\theta^j} \frac{P_{(j+1)q-l-j}(s)}{\{\prod_{k=1}^{q}(s + is_k)\}^{j+1}}, \quad j = 0, 1, \cdots; \quad l = 1, 2, \cdots, q, \tag{4.32}$$

where as before $P_m(s)$ denotes a polynomial of degree at most m. In a manner similar to that leading to (3.14) and (3.15), one obtains

$$\beta_{j,l} = -\sum_{j'=1}^{q-l} \sum_{l'=l+j'}^{q} b_{j',l,l'} \theta^{-j'} \beta_{j-j',l'} + \frac{1}{2\pi i} \int_{\mathcal{C}} A_{j,l}(s,\theta) h_0(s,\theta) ds, \tag{4.33}$$

for $j = 0, 1, \cdots$, and $l = 1, 2, \cdots, q$, where $\beta_{k,l'} \equiv 0$ for $k < 0$ and each

$$b_{j',l,l'} = \frac{1}{2\pi i} \int_{\mathcal{C}} \left\{ \theta^{j'} A_{j',l}(s,\theta) \right\} s^{l'-1} ds$$

is a constant independent of θ, s and the integration path $\mathcal{C}$ since this quantity depends only on the behavior of $\theta^{j'} A_{j',l}(s,\theta)$ (which is independent of θ; cf. (4.32) above) as $s \to \infty$.

We close this subsection by introducing another class of rational functions. Let us define inductively

$$Q_0(u,s,\theta) = \frac{1}{u-s}, \tag{4.34}$$

and

$$Q_{j+1}(u,s,\theta) = \frac{1}{\theta} \left[\frac{d}{du} + \sum_{l=1}^{q} \frac{\alpha_l}{u + i s_l} \right] Q_j(u,s,\theta) \tag{4.35}$$

for $j = 0, 1, 2, \cdots$. As in (4.32), we have

$$Q_j(u,s,\theta) = \frac{1}{\theta^j} \sum_{l=0}^{j} \frac{P_{l(q-1)}(u)}{(u-s)^{j-l+1} \left\{ \prod_{k=1}^{q} (u + i s_k) \right\}^l} \tag{4.36}$$

for $j = 0, 1, 2, \cdots$, where $P_m(u)$ denotes a polynomial of degree at most m. Furthermore, from (4.36) one gets

$$Q_j(u,s,\theta) = \frac{1}{\theta^j} \cdot \frac{1}{u^{j+1}} \sum_{k=0}^{\infty} \frac{P_{j,k}(s)}{u^k} \tag{4.37}$$

for fixed s and large u, where again $P_{j,k}(s)$ denotes a polynomial of degree at most k, whose coefficients are independent of u, s and θ.

Now assume that $\mathcal{C}$ is a simple closed u-contour that encloses $u = s$ and the points $u = -s_l$ for $l = 1, 2, \cdots, q$. Since (4.37) suggests that

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \theta^{j'} u^{l-1} Q_{j'}(u,s,\theta) du = \begin{cases} P_{j',l-j'-1}(s), & \text{if } l - j' - 1 \geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.38}$$

on account of (4.14) and (4.15) we obtain the following expression

$$\begin{aligned} h_j(s,\theta) &= \frac{1}{2\pi i}\int_{\mathcal{C}} Q_0(u,s,\theta)h_j(u,\theta)du \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} Q_1(u,s,\theta)h_{j-1}(u,\theta)du - \sum_{l=2}^{q}\beta_{j-1,l}(\theta)\cdot\theta^{-1}P_{1,l-2}(s) \\ &\cdots\cdots \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} Q_j(u,s,\theta)h_0(u,\theta)du - \sum_{j'=1}^{q-1}\sum_{l=j'+1}^{q}\beta_{j-j',l}(\theta)\cdot\theta^{-j'}P_{j',l-j'-1}(s) \end{aligned} \tag{4.39}$$

for $j = 0, 1, 2, \cdots$.

4.3. Properties of $T_l(x)$. To investigate the behavior of the functions $T_l(x)$ defined in (4.7) for bounded x or as $x \to +\infty$, we consider the integral

$$I_{f,\alpha}(x) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} f(s)s^{-\alpha}e^{xs}ds, \tag{4.40}$$

where $f(s)$ is an analytic function in $|\operatorname{Im} s| < \delta_0$ and satisfies

$$|f(s)| \le Ce^{K|\operatorname{Re} s|}$$

for $\operatorname{Re} s \le M_0$ and $|\operatorname{Im} s| \le \delta_0 - \epsilon$, and C, K, δ_0 and ϵ $(< \delta_0)$ are constants. Under these conditions, we have

$$I_{f,\alpha}(x) \sim \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(\alpha-k)}x^{\alpha-k-1}, \qquad \text{as} \quad x \to +\infty, \tag{4.41}$$

where the coefficients c_k are given by the Maclaurin expansion

$$f(s) = \sum_{k=0}^{\infty} c_k s^k,$$

which holds in a disc of radius at least δ_0. For a proof of (4.41), the reader is referred to Wong [13, p. 48, Ex.14].

We now return to $T_l(x)$ given in (4.7). For each factor $(s + is_k)^{-\alpha_k}$ in (4.7), we choose the branch which is positive when $s + is_k$ is positive. Then we deform the path of integration Γ and break it into q portions: Γ_1, Γ_2, $\cdots$, Γ_q. If $l > 1$ and $s_k \ne 0$ for $k = 1, \cdots, q$, then we set $s_0 = 0$ and introduce an additional portion Γ_0. Here, each Γ_k is a loop which starts at $-\infty - is_k$, encircles $s = -is_k$ in the counterclockwise direction, and ends at $-\infty - is_k$. Each Γ_k is carefully chosen so that it encloses no other branch points $s = -is_{k'}$, $k' \ne k$.

As an illustration, we consider T_1, and put $f_k(s) \equiv \prod_{k' \neq k} (s + i(s_{k'} - s_k))^{-\alpha_{k'}}$. From (4.7), we have

$$T_1(x) = \sum_{k=1}^{q} \frac{1}{2\pi i} \int_{\Gamma_k} f_k(s + is_k)(s + is_k)^{-\alpha_k} e^{xs} ds. \tag{4.42}$$

In each integral on the right-hand side of (4.42), we make a translation, and apply (4.41). The result is

$$\begin{aligned} T_1(x) &= \sum_{k=1}^{q} e^{-is_k x} I_{f_k, \alpha_k}(x) \\ &\sim \sum_{k=1}^{q} e^{-is_k x} \sum_{l=0}^{\infty} \frac{c_{k,l}}{\Gamma(\alpha_k - l)} x^{\alpha_k - l - 1} \end{aligned} \tag{4.43}$$

as $x \to +\infty$, where $c_{k,l}$ is the coefficient in the Maclaurin expansion

$$f_k(s) = \sum_{l=0}^{\infty} c_{k,l} s^l.$$

Since $s_k \neq s_{k'}$ for $k \neq k'$, and since $c_{k,0} = \prod_{k' \neq k} (is_{k'} - is_k)^{-\alpha_{k'}} \neq 0$, if each α_k is not an integer, then from (4.43) it follows that there are constants M and C such that

$$|T_1(x)| \geq C x^{\alpha^* - 1} \qquad \text{for} \quad x \geq M, \tag{4.44}$$

where $\alpha^* = \max_k \{\alpha_k\}$.

When $l > 1$, there are two subcases to be considered: (i) $s_k \neq 0$ for all $k = 1, \cdots, q$; and (ii) $s_k = 0$ for some k. The argument given above applies to both cases. Indeed, we have

$$T_l(x) \sim \sum_{k=1}^{q} e^{-is_k x} \sum_{l=0}^{\infty} \frac{\tilde{c}_{k,l}}{\Gamma(\alpha_k - l)} x^{\alpha_k - l - 1} \tag{4.45}$$

as $x \to +\infty$, when $s_k \neq 0$ for $k = 1, 2, \cdots, q$ where $\tilde{c}_{k,l}$ is the coefficient of s^l in the Maclaurin expansion

$$(s - is_k)^{l-1} f_k(s) = (s - is_k)^{l-1} \sum_{l=0}^{\infty} c_{k,l} s^l = \sum_{l=0}^{\infty} \tilde{c}_{k,l} s^l$$

and $\tilde{c}_{k,0} = (-is_k)^{l-1} c_{k,0} \neq 0$. Also, we have

$$\begin{aligned} T_l(x) &= \sum_{k=1}^{q} e^{-is_k x} I_{\hat{f}_k, \hat{\alpha}_k}(x) \\ &\sim \sum_{k=1}^{q} e^{-is_k x} \sum_{l=0}^{\infty} \frac{\hat{c}_{k,l}}{\Gamma(\hat{\alpha}_{k,l} - l)} x^{\hat{\alpha}_k - l - 1} \end{aligned} \tag{4.46}$$

as $x \to +\infty$, where $\hat{\alpha}_k = \alpha_k$ if $s_k \neq 0$, $\hat{\alpha}_k = \alpha_k - (l-1)$ if $s_k = 0$, and $\hat{c}_{k,l}$ is the coefficient in the Maclaurin expansion

$$\hat{f}_k(s) = \prod_{k' \neq k} (s + i(s_{k'} - s_k))^{-\hat{\alpha}_k} = \sum_{l=0}^{\infty} \hat{c}_{k,l} s^l;$$

in particular, $\hat{c}_{k,0} = \prod_{k' \neq k} (is_{k'} - is_k)^{-\hat{\alpha}_k} \neq 0$.

Now, let us consider $T_l(x)$ when x is bounded. We again start with (4.7), and choose Γ_0 so that $|s_k/s| < 1$ for $k = 1, 2, \cdots, q$ and $s \in \Gamma_0$. Clearly, we can write

$$\prod_{k=1}^{q} (s + is_k)^{-\alpha_k} = s^{-\alpha} \prod_{k=1}^{q} \left(1 + \frac{is_k}{s}\right)^{-\alpha_k} = s^{-\alpha} \sum_{k=0}^{\infty} a_k s^{-k},$$

where a_k can be expressed in terms of s_k and α_k, and $a_0 = 1$. Convergence of the last expansion is absolute and uniform for $s \in \Gamma$. Inserting this into (4.7) gives

$$T_l(x) = \sum_{k=0}^{\infty} a_k \frac{1}{2\pi i} \int_{\Gamma} s^{l-\alpha-k-1} e^{xs} ds = x^{\alpha - l} \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(\alpha - l + k + 1)} x^k. \tag{4.47}$$

From (4.47) it is evident that each $x^{l-\alpha} T_l(x)$ is an entire function, and that the behavior of $T_l(x)$ as $x \to 0$ is $x^{\alpha - l}$. Furthermore, there exists a small $\varepsilon > 0$ such that

$$\sum_{l=1}^{q} |T_l(x)| \geq C x^{\alpha - q} \tag{4.48}$$

for $x \in (0, \epsilon]$, where C is a positive constant. Next, we wish to extend the interval of validity of (4.48) to $x \in (0, M]$ for any finite M. To this aim, we only need to show that $T_l(x)$, $l = 1, 2, \cdots, q$, have no common zero in $(0, +\infty)$. This can be shown by the fact that

$$T_l(x) = \frac{d^{l-1}}{dx^{l-1}} T_1(x), \quad l = 1, 2, \cdots, q, \tag{4.49}$$

and that $T_1(x)$ satisfies an ordinary differential equation in x of order q. Indeed, we can write

$$\prod_{k=1}^{q} (s + is_k) = \sum_{l=0}^{q} d_l s^l,$$

where $d_q = 1$. Hence,

$$\frac{1}{2\pi i} \int_{\Gamma} \prod_{k=1}^{q} (s + is_k)^{1-\alpha_k} e^{xs} ds = \sum_{l=0}^{q} d_l \frac{d^l}{dx^l} T_1(x). \tag{4.50}$$

Using integration by parts once, it can be shown that the left-hand side of the last equation is equal to

$$-\frac{1}{x} \frac{1}{2\pi i} \int_{\Gamma} P_{q-1}(s) \prod_{k=1}^{q} (s + is_k)^{-\alpha_k} e^{xs} ds, \tag{4.51}$$

where

$$P_{q-1}(s) = \sum_{k=1}^{q}(1-\alpha_k)\prod_{k'\neq k}(s+is_k) = \sum_{l=0}^{q-1} c_l s^l.$$

In view of (4.50), inserting the last equation into (4.51) eventually leads to

$$x\frac{d^q}{dx^q}T_1(x) + \sum_{l=0}^{q-1}(xd_l + c_l)\frac{d^l}{dx^l}T_1(x) = 0. \tag{4.52}$$

Hence, if there exists some $x_0 > 0$ such that

$$\frac{d^l}{dx^l}T_1(x_0) \equiv T_{l+1}(x_0) = 0, \quad l = 0, 1, \cdots, q-1,$$

then from (4.52) we conclude that

$$\frac{d^l}{dx^l}T_1(x_0) = 0 \qquad \text{for } l = 0, 1, \cdots$$

Since $T_1(x)$ is analytic, it follows that $T_1(x) \equiv 0$. This contradicts (4.47), and implies that on the positive half real line there exists no common zero of $T_1(x)$, $T_2(x)$, $\cdots$, and $T_q(x)$. Consequently, (4.48) holds in $0 < x \leq M$ for any finite M.

4.4. Uniform asymptotic expansion. With all the preliminary preparation done for the special case (4.21), we can now establish that the formal expansion (4.16) is in fact a uniform asymptotic expansion.

THEOREM 2. *Let $a_n(\theta)$ and $z_k(\theta)$ be defined as in (4.1). If $z_k(\theta) = e^{is_k\theta}$ for some constants s_k then for $\theta \in [0, \nu], \nu = \min_{1\leq k\leq q}\{\pi/|s_k|\}$, we have*

$$a_n(\theta) = \theta^{1-\alpha}\sum_{l=1}^{q} T_l(n\theta)\sum_{k=0}^{m-1}\frac{\beta_{k,l}(\theta)}{n^k} + \varepsilon(\theta, m), \tag{4.53}$$

where the coefficients $\beta_{k,l}(\theta)$ are given iteratively by (4.6), (4.14) and (4.15), and the error term $\varepsilon(\theta, m)$ is expressed explicitly in (4.10), (4.17)-(4.20). Furthermore, there exist constants M_k such that

$$|\beta_{k,l}(\theta)| \leq M_k|\theta|^{l-1} \tag{4.54}$$

for $l = 1, 2, \cdots, q$ and $k = 0, 1, \cdots$, and

$$|\varepsilon(\theta, m)| \leq M_m\frac{\theta^{1-\alpha}}{n^m}\sum_{l=1}^{q}|T_l(n\theta)|, \tag{4.55}$$

for $m = 1, 2, \cdots$. Here $\alpha = \sum_{k=1}^{q}\alpha_k$, M_k is a positive constant, independent of θ for $\theta \in [0, \pi-\delta], \delta > 0$, and $T_l(n\theta)$ is defined in (4.7).

The proof of this theorem is essentially the same as that given in the previous section, and the discussion is always divided into two cases: i) $n\theta$ is bounded, and ii) $n\theta \to +\infty$, with (4.48) and (4.44) replacing (3.22) and (3.23), respectively.

Fields' case (cf. [6] and equation (1.2) in this paper) takes place when

$$q = 3; \quad s_1 = 0, \ \alpha_1 = \lambda; \quad s_2 = 1, \ \alpha_2 = \Delta; \quad s_3 = -1, \ \alpha_3 = \Delta, \tag{4.56}$$

which is of course a special case of Theorem 2. The integrals $T_l(x)$ in this case are given by

$$T_l(x) = \frac{1}{2\pi i} \int_\Gamma s^{l-\lambda-1}(s^2+1)^{-\Delta} e^{xs} ds, \quad l = 1, 2 \text{ and } 3,$$

and they can be expanded into series of the form

$$\frac{x^{\lambda+2\Delta-l}}{\Gamma(\lambda+2\Delta-l+1)} \sum_{k=0}^{\infty} \frac{(\Delta)_k \left(-x^2/4\right)^k}{((\lambda+2\Delta-l+1)/2)_k \left((\lambda+2\Delta-l+2)/2\right)_k k!}.$$

In terms of the generalized hypergeometric function, we have

$$T_l(x) = \frac{x^{\lambda+2\Delta-l}}{\Gamma(\lambda+2\Delta-l+1)} \, {}_1F_2\left(\Delta; \frac{\lambda+2\Delta-l+1}{2}, \frac{\lambda+2\Delta-l+2}{2}; -\frac{x^2}{4}\right) \tag{4.57}$$

for $l = 1, 2$, and 3.

5 EXAMPLES AND EXTENSIONS

Many known orthogonal polynomials can be defined in terms of their generating functions. If the generating function happens to be of the form given in (2.1) or (4.1), then the results presented in this paper are of course immediately applicable. As a simple illustration, we consider the ultraspherical polynomial $P_n^{(\lambda)}(x)$ defined by

$$\left[(e^{i\theta} - z)(e^{-i\theta} - z)\right]^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(\cos\theta) z^n. \tag{5.1}$$

Clearly, Corollary 1 applies here, with $\alpha = \lambda$ and $a_n(\theta) = P_n^{(\lambda)}(\cos\theta)$. In view of (3.7) and Lemma 1, the coefficients $\widetilde{\alpha}_k(\theta)$ and $\widetilde{\beta}_k(\theta)$ in the asymptotic expansion (3.6) can be written as

$$\widetilde{\alpha}_k(\theta) = \frac{\sqrt{\pi}}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_C A_k(s,\theta) h_0(s,\theta) ds \tag{5.2}$$

and

$$\widetilde{\beta}_k(\theta) = -\frac{\sqrt{\pi}}{\Gamma(\alpha)} \frac{1}{2\pi i} \int_C B_k(s,\theta) h_0(s,\theta) ds, \tag{5.3}$$

$k = 0, 1, 2, \cdots$, where $\mathcal{C}$ is as before a closed contour enclosing $s = \pm i$ (see Figure 2). The rational functions $A_k(s,\theta)$ and $B_k(s,\theta)$ are given in (3.11) - (3.13); in particular, we have

$$\begin{aligned} A_0(s,\theta) &= \frac{s}{s^2+1}, & B_0(s,\theta) &= \frac{1}{s^2+1}, \\ A_1(s,\theta) &= \frac{1}{\theta}\left\{\frac{1}{s^2+1} + \frac{2(\alpha-1)s^2}{(s^2+1)^2}\right\}, & B_1(s,\theta) &= \frac{1}{\theta}\left\{\frac{2(\alpha-1)s}{(s^2+1)^2}\right\}. \end{aligned} \tag{5.4}$$

Since $f(z,\theta) \equiv 1$ in (2.1), the function $h_0(s,\theta)$ in (2.9) becomes

$$h_0(s,\theta) = \left[\left(\frac{e^{-s\theta} - e^{i\theta}}{(-s-i)\theta}\right)\left(\frac{e^{-s\theta} - e^{-i\theta}}{(-s+i)\theta}\right)\right]^{-\lambda}. \tag{5.5}$$

Replacing α by λ in (5.2) and (5.3), we obtain by straightforward calculation

$$\widetilde{\alpha}_0(\theta) = \frac{\sqrt{\pi}}{\Gamma(\lambda)}\left(\frac{\sin\theta}{\theta}\right)^{-\lambda}\cos\lambda\theta, \qquad \widetilde{\beta}_0(\theta) = -\frac{\sqrt{\pi}}{\Gamma(\lambda)}\left(\frac{\sin\theta}{\theta}\right)^{-\lambda}\sin\lambda\theta,$$

$$\widetilde{\alpha}_1(\theta) = \frac{\sqrt{\pi}}{\Gamma(\lambda)}\lambda\left(\frac{\sin\theta}{\theta}\right)^{-\lambda}\left\{\frac{\lambda-1}{2}\left[-\frac{\theta\cos\theta - \sin\theta}{\theta\sin\theta}\sin\lambda\theta + 2\cos\lambda\theta\right] + \frac{\sin\lambda\theta}{\theta}\right\}$$

and

$$\widetilde{\beta}_1(\theta) = -\frac{\sqrt{\pi}}{\Gamma(\lambda)}\frac{1}{2}\lambda(\lambda-1)\left(\frac{\sin\theta}{\theta}\right)^{-\lambda}\left[\frac{\theta\cos\theta - \sin\theta}{\theta\sin\theta}\cos\lambda\theta + 2\sin\lambda\theta\right].$$

As a second example, suggested in Fields [6], we consider the polynomials

$$S_n^{(\lambda)}(x) = \sum_{k=0}^{n} P_k^{(\lambda)}(x). \tag{5.6}$$

The generating function of these polynomials is given by

$$(1-z)^{-1}[(e^{i\theta} - z)(e^{-i\theta} - z)]^{-\lambda} = \sum_{n=0}^{\infty} S_n^{(\lambda)}(\cos\theta)z^n \tag{5.7}$$

for $|z| < 1$. This is clearly a special case of (1.2), and Theorem 2 applies with $a_n(\theta) = S_n^{(\lambda)}(\cos\theta), q = 3, s_1 = 0, s_2 = 1$ and $s_3 = -1$. Furthermore, we have $\alpha_1 = 1, \alpha_2 = \alpha_3 = \lambda$ and $f(z,\theta) \equiv 1$. Hence, $\alpha = 1 + 2\lambda$ in (4.53). The approximants $T_l(n\theta), l = 0, 1, 2$, can be expressed in terms of the Gauss hypergeometric function, as is done in (4.57). To evaluate the coefficients $\beta_{k,l}(\theta)$ in the asymptotic expansion, we first note that the function $h_0(s,\theta)$ in (4.6) is given by

$$h_0(s,\theta) = \left(\frac{1-e^{-s\theta}}{s\theta}\right)^{-1}\left(\frac{e^{i\theta} - e^{-s\theta}}{i\theta + s\theta}\right)^{-\lambda}\left(\frac{e^{-i\theta} - e^{-s\theta}}{-i\theta + s\theta}\right)^{-\lambda}. \tag{5.8}$$

Also, from (4.30) we have

$$A_{0,1}(s,\theta) = \frac{1}{s}, \qquad A_{0,2}(s,\theta) = \frac{1}{s^2+1}, \qquad A_{0,3}(s,\theta) = \frac{1}{s(s^2+1)}.$$

Moreover, it follows from (4.29) that

$$A_{1,1}(s,\theta) = \frac{1}{\theta}\frac{2\lambda}{s^2+1}, \qquad A_{1,2}(s,\theta) = \frac{1}{\theta}\frac{(2\lambda-1)s^2+1}{s(s^2+1)^2}, \qquad A_{1,3}(s,\theta) = \frac{1}{\theta}\frac{2(\lambda-1)}{(s^2+1)^2}.$$

Using (4.28), we obtain

$$\beta_{0,1}(\theta) = \left(\frac{\sin\theta/2}{\theta/2}\right)^{-2\lambda}, \quad \beta_{0,2}(\theta) = \left(\frac{\sin\theta}{\theta}\right)^{-\lambda}\left(\frac{\theta/2}{\sin\theta/2}\right)\sin(\lambda+\frac{1}{2})\theta$$

and

$$\beta_{0,3}(\theta) = \left(\frac{\sin\theta/2}{\theta/2}\right)^{-2\lambda} - \left(\frac{\sin\theta}{\theta}\right)^{-\lambda}\left(\frac{\theta/2}{\sin\theta/2}\right)\cos(\lambda+\frac{1}{2})\theta.$$

The next set of coefficients $\beta_{1,l}(\theta), l = 1,2,3$, can be derived from (4.33), and they are given by

$$\begin{aligned} \beta_{1,1}(\theta) &= -\, b_{1,1,2}\,\theta^{-1}\beta_{0,2}(\theta) - b_{1,1,3}\,\theta^{-1}\beta_{0,3}(\theta) + \frac{1}{2\pi i}\int_{\mathcal{C}} A_{1,1}(s,\theta)h_0(s,\theta)ds, \\ \beta_{1,2}(\theta) &= -b_{1,2,3}\,\theta^{-1}\beta_{0,3}(\theta) + \frac{1}{2\pi i}\int_{\mathcal{C}} A_{1,2}(s,\theta)h_0(s,\theta)ds, \\ \beta_{1,3}(\theta) &= \frac{1}{2\pi i}\int_{\mathcal{C}} A_{1,3}(s,\theta)h_0(s,\theta)ds, \end{aligned} \tag{5.9}$$

where $\mathcal{C}$ is a simple closed curve enclosing $s = 0$ and $s = \pm i$. From the formula following (4.33), we get

$$b_{1,1,2} = 2\lambda, \qquad b_{1,1,3} = 0, \qquad b_{1,2,3} = 2\lambda - 1.$$

The three integrals in (5.9) can be evaluated by using residue theory (or Maple), and the results are very complicated. However, straightforward calculation shows that $\beta_{1,1}(\theta) \equiv 0$, and one can verify that the first few coefficients in the Maclaurin expansions of $\beta_{1,2}(\theta)$ and $\beta_{1,3}(\theta)$ vanish. Indeed, we have $\beta_{1,2}(\theta) = \sum_{k=1}^{\infty} *\theta^k$ and $\beta_{1,3}(\theta) = \sum_{k=2}^{\infty} *\theta^k$, thus showing that the estimates in (4.54) hold for $k = 0$ and $k = 1$.

In Darboux's original treatment, the singularities are fixed. Hence, only those on the circle of convergence are of importance. Contributions from the branch points and poles outside the circle of convergence are exponentially small, in comparison with those from the points on the circle. However, the situation is changed, when one considers uniform asymptotic expansions. Singularities not on the circle of convergence may become relevant, when they are allowed to approach the circle. Therefore, more general settings can be considered. (This has been suggested by one of the referees.) For example, instead of the branch points at $z = e^{\pm i\theta}$ in (2.1), one may consider the case where the nearest singularities are located at $z = 1 \pm i\theta$ with $\theta \in [0, \rho]$. More generally, we may relax the restriction $|z_k(\theta)| = 1$ in (4.1), and write $z_k(\theta) = e^{i\theta s_k(\theta)}$ with $s_k(\theta)$ being allowed to be complex. For instance, if $z_k(\theta) = 1 + d_k\theta$, then we may write $s_k(\theta) = -\frac{i}{\theta}\log(1 + d_k\theta)$ and this function is analytic in θ. The analysis in Section 4 continues to hold, and one readily verifies that the function $h_0(s, \theta)$ in (4.6) is still analytic in s and uniformly bounded in a s-domain of size $O\big(\frac{1}{\theta}\big)$. Of course, some of the formulas and arguments need minor modifications. For example, the coefficients of the polynomials in (4.32) are now analytic functions of θ, and so are the coefficients $b_{j',l,l'}$ in (4.33). However, the definition of $T_l(x)$ remains the same. Since all differentiations and integrations are with respect to s, analytic functions of θ can be treated as constants, and most of the derivations and estimations given in this paper can be extended to the more general setting mentioned above.

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